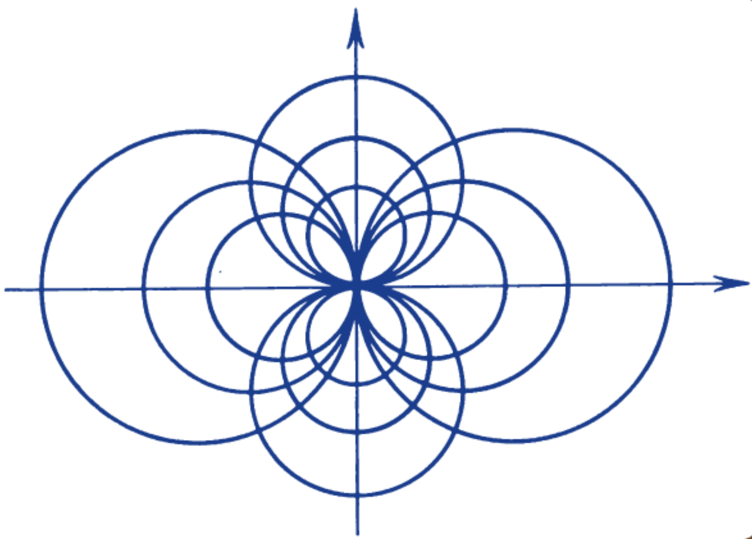
M.L. KRASNOV A.I. KISELYOV G.I. MAKARENKO

A BOOK OF PROBLEMS

IN ORDINARY DIFFERENTIAL EQUATIONS



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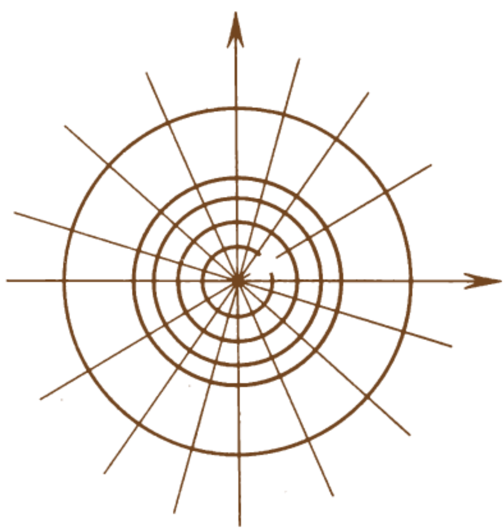
This problem book contains exercises for courses in differential equations at technical institutes.

Topics covered include, *inter alia,* the method of isoclines for equations of the first and second order, problems in finding orthogonal trajectories, the use of the method of superposition in solving linear differential equations of order n, linear dependence and linear independence of a system of functions, problems in solving linear equations with constant and variable coefficients, boundary value problems for different equations, integrating equations in power series, asymptotic integration, integrating systems of differential equations, Lyapunov stability, and the operator method.

Each section begins with a summary of basic facts and comprises worked-out examples of typical problems.

The book contains 967 problems, a considerable proportion of which have been composed by the authors themselves. There are some exercises of a theoretical nature.

The book is intended for students of technical institutes.



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ИЗДАТЕЛЬСТВО · МИР ·

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А. И. КИСЕЛЁВ,
Г. И. МАКАРЕНКО

СБОРНИК ЗАДАЧ ПО ОБЫКНОВЕННЫМ ДИФФЕРЕНЦИАЛЬНЫМ УРАВНЕНИЯМ

ИЗДАТЕЛЬСТВО «ВЫСШАЯ ШКОЛА» МОСКВА

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A.I. KISELYOV,
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A BOOK OF PROBLEMS

IN ORDINARY DIFFERENTIAL EQUATIONS

Translated from
the Russian
by Vladimir Shokurov

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На английском языке

PREFACE TO THE THIRD RUSSIAN EDITION

The third edition of this book is to a considerable extent rewritten and enlarged. Many problems have been substituted for by new ones; some problems with cumbersome solutions have been excluded; more than 50 examples worked out in the text have been added; misprints and inaccuracies in statements that were noticed have been eliminated. The most substantial additions relate to the following questions:
(a) solution of systems of differential equations;
(b) investigation of Lyapunov stability of solutions;
(c) use of the method of superposition in solving linear differential equations of order n; (d) asymptotic integration.

For the ease of using the book a special sign (◆) denoting that the solution of an example or the statement of a remark is over is sometimes used.

In the preparation of the book for publication a great help was rendered to us, in the capacity of reviewers of the manuscript, by Professor B. A. Bogatov and Assistant Professor A. I. Shum, of the Kalinin Polytechnical Institute, and by the workers of the Higher Mathematics Department, Moscow Institute of Electronics Engineering, headed by Professor A. V. Yefimov. We express our sincere gratitude to them. We are also obliged to N. N. Zarubina for her labour in preparing the drawings.

Although this is the third edition of the book, we are aware that it cannot be entirely faultless. We shall gratefully accept any criticisms and suggestions for its improvement.

The authors.

FROM THE PREFACE TO THE SECOND RUSSIAN EDITION

This problem book contains exercises for courses in differential equations at technical institutes.

Particular attention is given to questions which are not sufficiently treated in the available textbooks and which, as is shown by experience, are poorly understood by students. Worked out in detail are the method of isoclines for equations of the first and the second order, problems in finding orthogonal trajectories, linear dependence and linear independence of systems of functions.

The book includes a large number of problems in solving linear equations with constant and variable coefficients, problems in Lyapunov stability, in applying the operator method to solving differential equations and systems.

Some new sections have been introduced: the method of successive approximations, particular solutions of differential equations, and equations with a small parameter of the derivative. The section devoted to the use of series in solving differential equations has been enlarged, some revisions have been made, and errors and misprints noticed in the first edition corrected.

CONTENTS

CHAPTER I

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

1. Basic concepts and definitions

A differential equation is an equation connecting an independent variable x, a sought-for function $y = y(x)$ and its derivatives $y', y'', \ldots, y^{(n)}$, i.e. an equation of the form

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0.$$

If the sought-for function $y = y(x)$ is a function of one independent variable x, the differential equation is called *ordinary**; for instance,

$$(1)\ \frac{dy}{dx} + xy = 0, \quad (2)\ y'' + y' + x = \cos x,$$

$$3)\ (x^2 - y^2)\,dx - (x + y)\,dy = 0.$$

When the sought-for function y is a function of two or more independent variables, if for example $y = y(x, t)$, then the equation of the form

$$F\left(x, t, y, \frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}, \ldots, \frac{\partial^m y}{\partial x^k \partial t^l}\right) = 0$$

is called *a partial differential equation*. Here k, l are nonnegative integers such that $k + l = m$; for instance,

$$\frac{\partial y}{\partial t} - \frac{\partial y}{\partial x} = 0, \quad \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}.$$

The order of a differential equation is the order of the highest derivative contained in the equation. Thus the differential equation $y' + xy = e^x$ is an equation of the first order, the differential equation $y'' + p(x)\,y = 0$, where $p(x)$ is a known function, is an equation of the second order; the

* Only ordinary differential equations will be considered in what follows.

differential equation $y^{(9)} - xy'' = x^2$ is an equation of the ninth order.

A solution of a differential equation of the nth order on the interval (a, b) is a function $y = \varphi(x)$ determined on the interval (a, b) together with its derivatives up to the nth order and such that substituting the function $y = \varphi(x)$ in the differential equation transforms the latter into an identity in x on (a, b). For instance, the function $y = \sin x + \cos x$ is a solution of the equation $y'' + y = 0$ on the interval $(-\infty, +\infty)$. Indeed, differentiating the function twice we shall have

$$y' = \cos x - \sin x, \quad y'' = -\sin x - \cos x.$$

Substituting the expressions for y'' and y in the differential equation we shall obtain the identity

$$-\sin x - \cos x + \sin x + \cos x \equiv 0.$$

The graph of a solution of a differential equation is called *an integral curve* of the equation.

The general form of an equation of the first order is

$$F(x, y, y') = 0. \tag{1}$$

If equation (1) can be solved for y', then one has

$$y' = f(x, y) \tag{2}$$

—*an equation of the first order solved for the derivative.*

The Cauchy, or initial value, problem is a problem of finding the solution $y = y(x)$ of the equation $y' = f(x, y)$ satisfying the initial condition $y(x_0) = y_0$ (or, in another writing, $y\,|_{x=x_0} = y_0$).

Geometrically this means that an integral curve passing through a given point $M_0(x_0, y_0)$ in the xOy plane (Fig. 1) is sought.

The existence and uniqueness theorem for the initial value problem. *Let a differential equation $y' = f(x, y)$ be given, where the function $f(x, y)$ is determined in some domain D in the xOy plane containing a point (x_0, y_0). If the function $f(x, y)$ satisfies the conditions*

(a) *$f(x, y)$ is a continuous function of two variables x and y in the domain D*;

(b) *$f(x, y)$ has a partial derivative $\partial f/\partial y$ bounded in the domain D, then an interval $(x_0 - h, x_0 + h)$ will be found*

on which there exists a unique solution $y = \varphi(x)$ of the given equation satisfying the condition $y(x_0) = y_0$.

The theorem gives sufficient conditions for the existence of a unique solution of the initial value problem for the

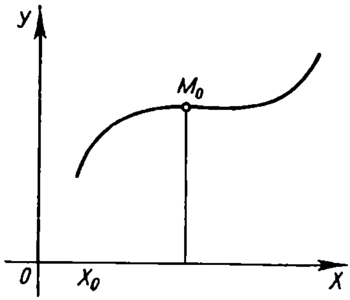

Fig. 1

equation $y' = f(x, y)$ but these conditions are not necessary. That is, there may exist a unique solution of the

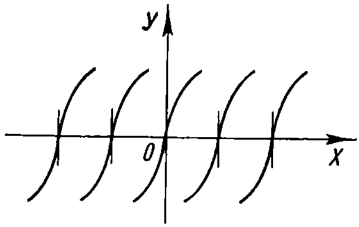

Fig. 2

equation $y' = f(x, y)$ satisfying the condition $y(x_0) = y_0$ although one of the conditions (a) or (b) or both are not satisfied at the point (x_0, y_0).

Consider some examples. 1. $y' = \frac{1}{y^2}$. Here $f(x, y) = 1/y^2$, $\partial f/\partial y = -2/y^3$. At the points $(x_0, 0)$ on the Ox axis conditions (a) and (b) are not satisfied (the function $f(x, y)$ and its partial derivative $\partial f/\partial y$ are discontinuous on the Ox axis and unbounded when $y \to 0$), but a unique integral curve $y = \sqrt[3]{3(x - x_0)}$ passes through each point on the Ox axis (Fig. 2). ◆

2. $y' = xy + e^{-y}$. The right-hand side of the equation $f(x, y) = xy + e^{-y}$ and its partial derivative $\partial f/\partial y = x - e^{-y}$ are continuous in x and y at all points in the xOy plane. By virtue of the existence and uniqueness theorem the domain in which the given equation has a unique solution is the entire xOy plane. ◆

3. $y' = \frac{3}{2}\sqrt[3]{y^2}$. The right-hand side of the equation $f(x, y) = \frac{3}{2}\sqrt[3]{y^2}$ is determined and continuous at each point in the xOy plane. The partial derivative $\frac{\partial f}{\partial y} = 1/\sqrt[3]{y}$ goes into infinity when $y = 0$, i.e. on the Ox axis, so that

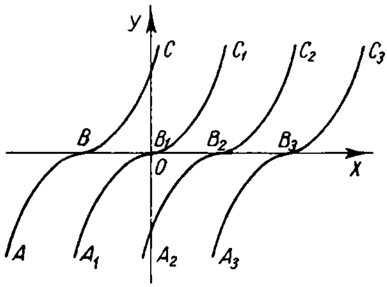

Fig. 3

when $y = 0$ condition (b) of the existence and uniqueness theorem is violated. Consequently, violations of uniqueness are possible at the points on the Ox axis. It is easy to verify that the function $y = (x + c)^3/8$ is a solution of the given equation. Besides, it has an obvious solution $y \equiv 0$. Thus at least two integral curves pass through each point on the Ox axis and hence uniqueness is in fact violated at the points of this axis (Fig. 3).

Integral curves of the given equation will also be curves made up of bits of cubic parabolas $y = (x + c)^3/8$ and x-intercepts, for instance $ABOC_1$, ABB_2C_2, A_2B_2x, etc., so that an infinite number of integral curves pass through each point on the Ox axis. ◆

Remark. The condition of boundedness of the derivative $\partial f/\partial y$ appearing in the existence and uniqueness theorem for the initial value problem may be somewhat weakened and replaced by the so-called *a Lipschitz condition.*

A function $f(x, y)$ defined in some domain D is said to satisfy in D a Lipschitz condition in y if there is a constant L (a Lipschitz constant) such that for any y_1, y_2 in D and any x in D the inequality

$$|f(x, y_2) - f(x, y_1)| \leqslant L |y_2 - y_1|$$

is valid.

The existence in the domain D of the bounded derivative $\partial f/\partial y$ is sufficient for a function $f(x, y)$ to satisfy in D a Lipschitz condition (prove this!). On the contrary, the condition of the boundedness of $\partial f/\partial y$ does not follow from a Lipschitz condition; the former may not even exist. For instance, for the equation $y' = 2|y|\cos x$ the function $f(x, y) = 2|y|\cos x$ is not differentiable with respect to y at the point $(x_0; 0)$, $x_0 \neq \frac{\pi}{2} + k\pi$, $k = 0, \pm 1, \ldots$, but the Lipschitz condition is in the neighbourhood of this point. Indeed

$$|f(x, y_2) - f(x, y_1)| = |2|y_2|\cos x - 2|y_1|\cos x|$$

$$= 2|\cos x|\,||y_2| - |y_1|| \leqslant 2|y_2 - y_1|,$$

since $|\cos x| \leqslant 1$. and $||y_2| - |y_1|| \leqslant |y_2 - y_1|$.

Thus the Lipschitz condition with the constant $L = 2$ is satisfied.

Theorem. *If the function $f(x, y)$ is continuous and satisfies a Lipschitz condition in y in the domain D, then the initial value problem*

$$\frac{\partial y}{\partial x} = f(x, y), \quad y|_{x=x_0} = y_0, \quad (x_0, y_0) \in D$$

has a unique solution.

A Lipschitz condition is essential for the uniqueness of the solution of an initial value problem. As an example consider an equation

$$\frac{\partial y}{\partial x} = f(x, y)$$

where

$$f(x, y) = \begin{cases} \dfrac{4x^3y}{x^4 + y^2}, & x^2 + y^2 > 0, \\ 0, & x = y = 0. \end{cases}$$

It can be easily seen that the function $f(x, y)$ is continuous; on the other hand,

$$f(x, Y)-f(x, y)=\frac{4x^3(x^4-yY)}{(x^4+y^2)(x^4+Y^2)}(Y-y).$$

If $y=\alpha x^2$, $Y=\beta x^2$, then

$$|f(x, Y)-f(x, y)|=\frac{4}{|x|}\left|\frac{1-\alpha\beta}{(1+\alpha^2)(1+\beta^2)}\right||Y-y|$$

and the Lipschitz condition is not satisfied in any of the domains containing the origin $O(0, 0)$ since the factor of $|Y-y|$ turns out to be unbounded when $x \to 0$.

The given differential equation allows the solution

$$y=C^2-\sqrt{x^4+C^4}$$

where C is an arbitrary constant. It follows that there are an infinite number of solutions satisfying the initial condition $y(0)=0$.

The general solution of the differential equation (2) is a function

$$y=\varphi(x, C) \tag{3}$$

depending on one arbitrary constant C and such that (a) it satisfies equation (2) with any allowed values of the constant C; (b) whatever the initial condition

$$y|_{x=x_0}=y_0 \tag{4}$$

it is possible to choose a value C_0 for the constant C such that the solution $y=\varphi(x, C_0)$ will satisfy the given initial condition (4). Here it is supposed that the point (x_0, y_0) lies in a domain where the existence and uniqueness conditions of the solution are satisfied.

A particular solution of the differential equation (2) is a solution obtained from the general solution (3) with some defined value of the arbitrary constant C.

Example 4. Verify that the function $y=x+C$ is the general solution of the differential equation $y'=1$ and find a particular solution satisfying the initial condition $y|_{x=0}=0$. Give a geometrical interpretation of the result.

Solution. The function $y=x+C$ satisfies the given equation with any values of the arbitrary constant C. Indeed $y'=(x+C)'=1$.

Let us specify an arbitrary initial condition $y\,|_{x=x_0} = y_0$. Putting $x = x_0$ and $y = y_0$ in the equation $y = x + C$ we find that $C = y_0 - x_0$. Substituting this value of C in the given function yields $y = x + y_0 - x_0$. This function satisfies the given initial condition: putting $x = x_0$ we obtain $y = x_0 + y_0 - x_0 = y_0$. So the function $y = x + C$ is the general solution of the given equation.

In particular, putting $x_0 = 0$ and $y_0 = 0$ we obtain a particular solution $y = x$.

The general solution of the given equation, i.e. the function $y = x + C$, determines in the xOy plane a family of

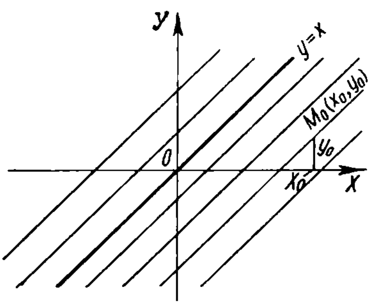

Fig. 4

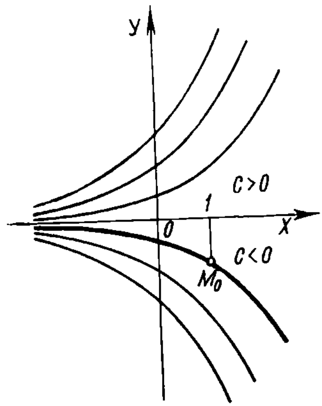

Fig. 5

parallel lines with a slope $k = 1$. A unique integral curve $y = x + y_0 - x_0$ passes through each point $M_0\,(x_0, y_0)$ of the xOy plane. The particular solution $y = x$ determines one of the integral curves, namely the straight line passing through the origin of coordinates (Fig. 4).

Example 5. Verify that the function $y = Ce^x$ is the general solution of the equation $y' - y = 0$ and find a particular solution satisfying the initial condition $y\,|_{x=1} = -1$.

Solution. We have $y = Ce^x$, $y' = Ce^x$. Substituting in the given equation the expressions for y and y' we obtain $Ce^x - Ce^x \equiv 0$, i.e. the function $y = Ce^x$ satisfies the given equation with any values of the constant C.

Specify an arbitrary initial condition $y\big|_{x=x_0} = y_0$. Substituting x_0 and y_0 for x and y in the function $y = Ce^x$ we have $y_0 = Ce^{x_0}$, whence $C = y_0e^{-x_0}$. The function $y = y_0e^{x-x_0}$ satisfies the initial condition. In fact, putting $x = x_0$ we obtain $y = y_0e^{x_0-x_0} = y_0$. The function $y = Ce^x$ is the general solution of the given equation.

When $x_0 = 1$ and $y_0 = -1$ we obtain a particular solution $y = -e^{x-1}$.

Geometrically, the general solution determines a family of integral curves which are the graphs of exponential functions; the particular solution is an integral curve passing through the point $M_0(1, -1)$ (Fig. 5). ◆

A relation of the form $\Phi(x, y, C) = 0$ implicitly determining the general solution is called *the general integral* of a differential equation of the first order.

A relation obtained from the general integral when the constant C has a particular value is called *a particular integral* of the differential equation.

A problem of solving or integrating a differential equation consists in finding the general solution or the general integral of a given differential equation. If in addition an initial condition is given, then it is required to separate out a particular solution or a particular integral satisfying the specified initial condition.

Since the x- and y-coordinates are geometrically equivalent, we shall consider the equation $\frac{dx}{dy} = \frac{1}{f(x, y)}$ along with the equation $\frac{dy}{dx} = f(x, y)$.

1. Find the coinciding solutions of the two equations:

(a) $y' = y^2 + 2x - x^4$; (b) $y' = -y^2 - y + 2x + x^2 + x^4$.

In the following problems separate out domains in which the given equations have unique solutions.

2. $y' = x^2 + y^2$.
3. $y' = \frac{x}{y}$.
4. $y' = y + 3\sqrt[3]{y}$.
5. $y' = \sqrt{x - y}$.
6. $y' = \sqrt{x^2 - y} - x$.
7. $y' = \sqrt{1 - y^2}$.
8. $y' = \frac{y+1}{x-y}$.
9. $y' = \sin y - \cos x$.
10. $y' = 1 - \cot y$.
11. $y' = \sqrt[3]{3x - y} - 1$.

12. Show that for the equation $y' = |y|^{1/2}$ the uniqueness of the solution is broken at each point on the Ox axis.

13. Find the integral curve of the equation $y' = \sin(x \cdot y)$ passing through the point O (0; 0).

In the following problems show that the given functions are solutions of the indicated differential equations:

14. $y = \frac{\sin x}{x}$, $xy' + y = \cos x$.

15. $y = Ce^{-2x} + \frac{1}{3}e^x$, $y' + 2y = e^x$.

16. $y = 2 + C\sqrt{1 - x^2}$, $(1 - x^2)y' + xy = 2x$.

2. The method of isoclines

The equation

$$y' = f(x, y) \tag{1}$$

determines at each point (x, y), at which there exists a function $f(x, y)$, the value of y', i.e. the slope of the tangent to the integral curve at that point.

If at each point of a domain D the value of a certain quantity is given, then *the field* of that quantity in the domain D is said to be given. Thus the differential equation (1) determines *a direction field*.

A triple of numbers $(x; y; y')$ determines the direction of a straight line passing through the point (x, y). The totality of the segments of these straight lines gives a geometrical representation of the direction field.

The problem of integrating the differential e uation (1) can now be interpreted like this: find the curve such that the tangent to it has at each point the direction coinciding with the direction of the field at that point.

The problem of constructing an integral curve is often solved by introducing isoclines. *An isocline* is a locus of points at which the tangents to the required integral curves differential equation (1) is determined by the equation

$$f(x, y) = k \tag{2}$$

where k is a parameter. Assigning to the parameter k close numerical values we get a sufficiently dense meshwork of isoclines by means of which it is possible to construct approximately the integral curves of the differential equation (1).

Remark 1. The zero isocline $f(x, y) = 0$ gives an equation of lines which may contain the extreme points of the integral curves. For a greater accuracy in constructing the integral curves the locus of the points of inflection is also found. To do this y'' is found by virtue of equation (1):

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' = \frac{\partial f}{\partial x} + f(x, y) \frac{\partial f}{\partial y} \tag{3}$$

and is equated to zero. The line determined by the equation

$$\frac{\partial f}{\partial x} + f(x, y) \frac{\partial f}{\partial y} = 0 \tag{4}$$

is a possible locus of the points of inflection.

Example 1. Construct approximately with the help of isoclines the integral curves of the differential equation $y' = 2x - y$.

Solution. To obtain the equation of isoclines put $y' = k$, $k = \text{const}$, then

$$2x - y = k \quad \text{or} \quad y = 2x - k.$$

Isoclines are parallel lines. With $k = 0$ we get the isocline $y = 2x$. This straight line bisects the xOy plane giving two parts in each of which the derivative y' has the same sign (Fig. 6).

The integral curves, cutting the straight line $y = 2x$, pass from the domain of the decrease of the function of y to the domain of its increase, and vice versa, which means that this straight line contains the extreme points of the integral curves, namely, the minimum points.

Take two more isoclines:

$$y = 2x + 1, \quad k = -1 \quad \text{and} \quad y = 2x - 1, \quad k = 1.$$

The tangents drawn to the integral curves at the points of intersection with the isoclines $k = -1$ and $k = 1$ make angles of 135° and 45° respectively with the Ox axis. Find then the second derivative $y'' = 2 - y' = 2 - 2x + y$.

The straight line $y = 2x - 2$ where $y'' = 0$ is an isocline obtained with $k = 2$ and at the same time an integral curve, which can be easily shown by substitution into the equation. Since the right-hand side of the given equation $f(x, y) = 2x - y$ satisfies the conditions of the existence and uniqueness theorem in the whole xOy plane, the other integral curves do not intersect this isocline. The isocline $y = 2x$

which contains the minimum points of the integral curves is over the isocline $y = 2x - 2$ and therefore the integral curves passing under the latter have no extreme points.

The straight line $y = 2x - 2$ divides the xOy plane into two parts in one of which (that over the straight line) $y'' > 0$ and hence the integral curves are concave upwards and in the other part $y'' < 0$ and hence the integral curves are

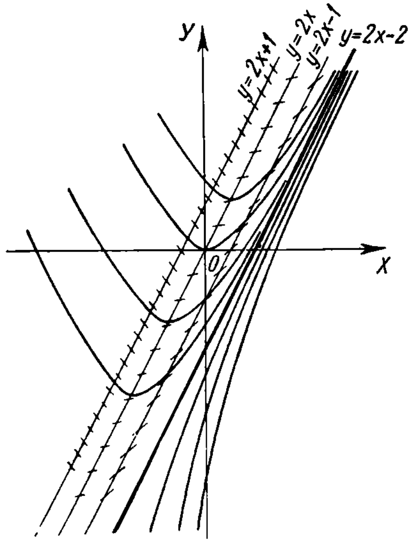

Fig. 6

concave downwards. No integral curves intersect the straight line $y = 2x - 2$, which means that it is not a locus of points of inflection. The integral curves of the given equation have no points of inflection.

The investigation we have carried out allows us to construct approximately a family of integral curves for the equation (Fig. 6).

Example 2. Using the method of isoclines construct approximately the integral curves of the differential equation $y' = \sin(x + y)$.

Solution. Putting $y' = k$, where $k = \text{const}$, we obtain the equation of isoclines $\sin(x + y) = k$, with $-1 \leqslant k \leqslant 1$.

When $k = 0$ we have $\sin(x + y) = 0$, whence

$$y = -x + \pi n \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{5}$$

The integral curves have horizontal tangents at the points of intersection with these isoclines.

Let us determine whether the integral curves have extreme points on the isoclines $y = -x + \pi n$. To do this we find the second derivative:

$$y'' = (1 + y') \cos(x + y) = [1 + \sin(x + y)] \cos(x+y).$$

When $y = -x + \pi n$ we have

$$y'' = (1 + \sin \pi n) \cos \pi n = \cos \pi n = (-1)^n.$$

If n is even, then $y'' > 0$, and hence the integral curves have minima at the points of intersection with the isoclines $y = -x + \pi n$, $n = 0, \pm 2, \pm 5, \ldots$; if, however, n is odd, then $y'' < 0$ and the integral curves have maxima at the points of intersection with the isoclines $y = -x + \pi n$, $n = \pm 1, \pm 3, \ldots$. We find the isoclines

$$k = -1, \sin(x + y) = -1; \quad y = -x - \frac{\pi}{2} + 2\pi n, \tag{6}$$

$$k = 1, \sin(x + y) = 1; \quad y = -x + \frac{\pi}{2} + 2\pi n, \tag{7}$$

$$n = 0, \pm 1, \pm 2, \ldots.$$

The isoclines are parallel lines with the slope equal to -1, i.e. they intersect the Ox axis at an angle of 135°. It is easy to see that the isoclines $y = -x - \frac{\pi}{2} + 2\pi n$, $n = 0$, $\pm 1, \ldots$ are the integral curves of the given differential equation (to do this it is sufficient to substitute the function $y = -x - \frac{\pi}{2} + 2\pi n$ in the equation $y' = \sin(x + y)$).

The right-hand side of this equation, i.e. the function $f(x, y) = \sin(x + y)$, satisfies all the conditions of the existence and uniqueness theorem at each point of the xOy plane and therefore the integral curves do not intersect and, consequently, they do not cut the isoclines $y = -x - \frac{\pi}{2} + 2\pi n$. Further the derivative y'' vanishes when $1 + \sin(x + y) = 0$, i.e. on the isoclines (6), and when $\cos(x + y) = 0$, i.e. on the isoclines (6) and (7). When going (from left to right) over the isoclines (7) y'' changes

the sign from plus to minus. For instance, if we consider the region contained between the isoclines $y = -x$ and $y = -x + \pi$, then on the isocline $y = -x + \frac{\pi}{2}$ the derivative $y'' = 0$, with $y'' > 0$ under the isocline. So the integral curves are concave upwards, and over the isocline $y'' < 0$ and so the integral curves are concave downwards. Thus the isoclines (7) are the locus of the points of inflection

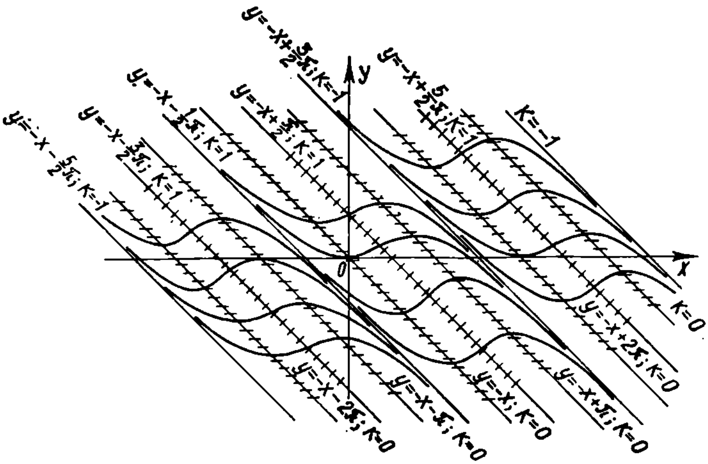

Fig. 7

of the integral curves. The data obtained make it possible to construct approximately the family of integral curves of the given equation. To obtain a more accurate construction one should draw some more isoclines (Fig. 7).

Example 3. Using the method of isoclines construct the integral curves of the equation $y' = y - x^2 + 2x - 2$.

Solution. Put $y' = k$, $k = \text{const}$. Then the equation of isoclines will take the form

$$y - x^2 + 2x - 2 = k \text{ or } y = x^2 - 2x + k.$$

The isoclines are parabolas with a vertical axis of symmetry $x = 1$. There are no integral curves among the isoclines. Indeed, substituting $y = x^2 - 2x + 2 + k$ and $y' = 2x - 2$ in the given equation we have $2x - 2 = x^2 - 2x + 2 + k - x^2 + 2x - 2$ or $2x - 2 = k$. But this equation cannot hold identically in x for any k.

Let $k = 0$. Then the integral curves will have horizontal tangents at the points of intersection with the isocline $y = x^2 - 2x + 2$. The isocline $y = x^2 - 2x + 2$ divides the xOy plane into two parts in one of which $y' < 0$ (solutions y decrease) and in the other $y' > 0$ (solutions y increase). And since this isocline is not an integral curve, it contains the extreme points of the integral curves, namely the minimum points in that part of the parabola $y = x^2 - 2x + 2$ where $x < 1$ and the maximum points in that part of the parabola where $x > 1$. The integral curve through the point (1; 1), i.e. through the vertex of the parabola $y = x^2 - 2x + 2$, has no extremum at this point. At the points of the isoclines $y = x^2 - 2x + 3$ ($k = 1$) and $y = x^2 - 2x + 1$ ($k = -1$) the tangents to the integral curves have the slopes equal to 1 and -1 respectively.

To investigate the direction of concavity in the integral curves, find the second derivative:

$$y'' = y' - 2x + 2 = y - x^2 + 2x - 2 - 2x + 2$$
$$= y - x^2.$$

It vanishes only at the points lying on the parabola $y = x^2$. At the points of the xOy plane whose coordinates satisfy the condition $y < x^2$ the integral curves are concave downward ($y'' < 0$) and at the points where $y > x^2$, they are concave upwards ($y'' > 0$). The points of intersection of integral curves with the parabola $y = x^2$ are the points of inflection of these curves. So the parabola $y = x^2$ is the locus of the points of inflection of the integral curves.

The right-hand side of the original equation $f(x, y) = y - x^2 + 2x - 2$ satisfies the conditions of the existence and uniqueness theorem at all points of the xOy plane, therefore a unique integral curve of the equation passes through each point of the plane.

Using the information obtained we construct approximately the family of integral curves of the given equation (Fig. 8).

Remark 2. The points of intersection of two or several isoclines may be *singular points of the differential equation* (1), i.e. such points at which the right-hand side of equation (1) is not determined.

Consider the equation $y' = y/x$. The family of isoclines is determined by the equation $y/x = k$. This is a family

of straight lines passing through the origin of coordinates, so that isoclines corresponding to different slopes of the tangents to the integral curves intersect at the origin. It can easily be seen that the general solution of the given equation is of the form $y = Cx$ and that the point (0, 0)

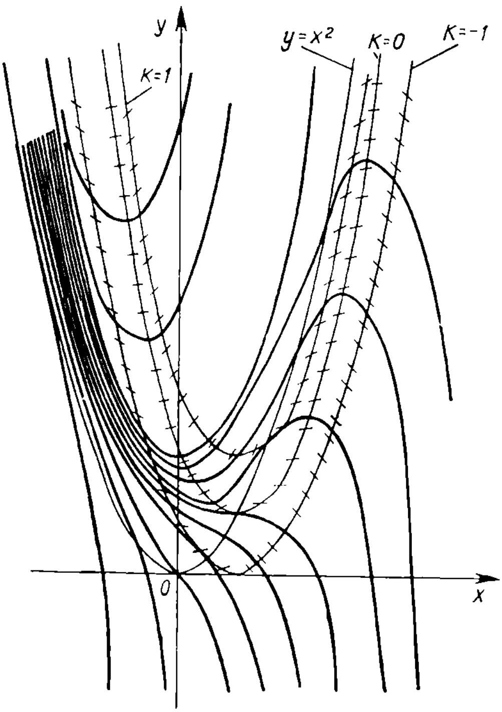

Fig. 8

is a singular point of the differential equation. Here the isoclines are the integral curves of the equation (Fig. 9).

Example 4. Using the method of isoclines construct the integral curves of the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$.

Solution. Putting $y' = k$, $k = \text{const}$, we obtain the equation of the family of isoclines $\frac{y-x}{y+x} = k$. Thus the isoclines are straight lines passing through the origin $O(0;0)$.

When $k = -1$ we get the isocline $y = 0$; when $k = 0$ we get the isocline $y = x$; and when $k = 1$ the isocline is $x = 0$.

Considering the "inverted" equation

$$\frac{dx}{dy} = \frac{y+x}{y-x}$$

we find the isocline $y = -x$ at all the points of which the integral curves have vertical tangents.

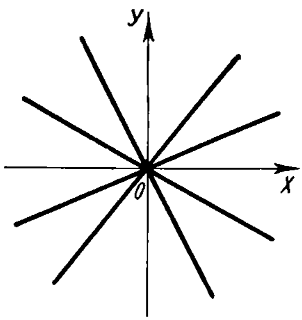

Fig. 9

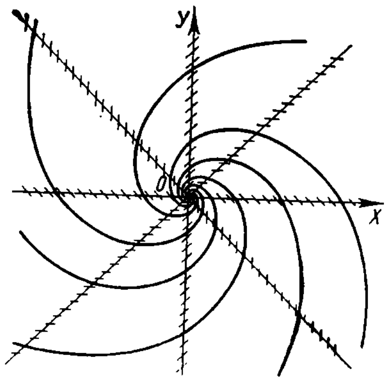

Fig. 10

The point (0; 0) is the point of intersection of all the isoclines of the given equation (the singularity of the equation).

Using the isoclines obtained we construct the integral curves (Fig. 10).

17. Find the angle α between the integral curves of the equations $y' = x + y$ and $y' = x - y$ at the point M (2, 1).

18. At what angle α do the integral curves of the equation $y' = x^2 + y^2 + 1$ intersect the Ox axis at the point O (0, 0)?

19. Find the extreme points of the integral curves of the equation $y' = x + 1$.

20. Find the points of inflection of the integral curves of the equation $y' = y - x^2$.

Using the method of isoclines construct the integral curves of the following differential equations:

21. $y' = x + 1.$

22. $y' = x + y.$

23. $y' = y - x.$

24. $y' = \frac{1}{2}(x - 2y + 3).$

25. $y' = (y - 1)^2.$

26. $y' = (y - 1)x.$

27. $y' = x^2 - y^2.$

28. $y' = \cos(x - y).$

29. $y' = y - x^2.$

30. $y' = x^2 + 2x - y.$

31. $y' = \frac{y+1}{x-1}.$

32. $y' = \frac{x+y}{x-y}.$

33. $y' = 1 - x.$

34. $y' = 2x - y.$

35. $y' = x^2 + y.$

36. $y' = -y/x.$

37. $y' = 1.$

38. $y' = 1/x.$

39. $y' = y.$

40. $y' = y^2.$

3. The method of successive approximations

Suppose it is required to find the solution $y = y(x)$ of the differential equation

$$y' = f(x, y) \tag{1}$$

satisfying the initial condition

$$y|_{x=x_0} = y_0. \tag{2}$$

Assume that in some rectangle

$$D\{|x - x_0| < a, \ |y - y_0| < b\}$$

with centre at the point (x_0, y_0) conditions (a) and (b) of the existence and uniqueness theorem for the problem (1)-(2) of Section 1, Chapter I, are satisfied for equation (1).

A solution of the problem (1)-(2) can be found by *the method of successive approximations* which is as follows.

We construct a sequence $\{y_n(x)\}$ of functions defined by the recurrence relations

$$y_n(x) = y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t)]\, dt, \quad n = 1, 2, \ldots . \tag{3}$$

As a zero-order approximation $y_0(x)$ we may take any function continuous in the neighbourhood of the point

$x = x_0$, in particular $y_0(x) \equiv y_0$, the Cauchy initial value (2). It is possible to prove that under the assumptions made concerning equation (1) the successive approximations $\{y_n(x)\}$ converge to an exact solution of equation (1) satisfying condition (2) in some interval $x_0 - h < x < x_0 + h$, where

$$h = \min\left(a, \frac{b}{M}\right), \quad M = \max_{(x, y)\in D} |f(x, y)|. \tag{4}$$

The estimate of the error resulting from the replacement of the exact solution $y(x)$ by the nth approximation $y_n(x)$ is given by the inequality

$$|y(x) - y_n(x)| \leqslant \frac{MN^{n-1}}{n!} h^n, \tag{5}$$

where $N = \max\limits_{(x, y)\in D} \left|\frac{\partial f}{\partial y}\right|$.

When using the method of successive approximations, one should choose such n for which $|y_{n+1} - y_n|$ does not exceed permissible error.

Example 1. Using the method of successive approximations find a solution of the equation $\frac{dy}{dx} = y$ satisfying the initial condition $y(0) = 1$.

Solution. It is obvious that for the given equation the conditions of the existence and uniqueness theorem for the initial value problem are satisfied in the whole xOy plane. We construct a sequence $\{y_n(x)\}$ of functions defined by relations (3) taking $y_0(x) \equiv 1$ as the zero-order approximation:

$$y_0(x) \equiv 1,$$

$$y_1(x) = 1 + \int_0^x y_0(t)\,dt = 1 + x,$$

$$y_2(x) = 1 + \int_0^x y_1(t)\,dt = 1 + \int_0^x (1 + t)\,dt = 1 + x + \frac{x^2}{2},$$

$$y_3(x) = 1 + \int_0^x y_2(t)\,dt = 1 + \int_0^x \left(1 + t + \frac{t^2}{2}\right) dt =$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

in general

$$y_n(x) = 1 + \int_0^x y_{n-1}(t)\,dt = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

It is clear that $y_n(x) \to e^x$ as $n \to \infty$. A direct check shows that the function $y(x) = e^x$ is the solution of the set initial value problem.

Example 2. Find by the method of successive approximations a solution of the equation $y' = x^2 + y^2$ satisfying the initial conditions $y|_{x=0} = 0$ in a rectangle $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$.

Solution. We have $|f(x, y)| = x^2 + y^2 \leqslant 2$, i.e. $M = 2$, h is taken to be the smaller of the numbers $a = 1$, $b/M = 1/2$, i.e. $h = 1/2$. According to (4) successive approximations will converge in the interval $-\frac{1}{2} < x < \frac{1}{2}$. We form them:

$$y_0(x) = 0,$$

$$y_1(x) = \int_0^x (t^2 + y_0^2)\,dt = \frac{x^3}{3},$$

$$y_2(x) = \int_0^x [t^2 + y_1^2(t)]\,dt = \int_0^x \left(t^2 + \frac{t^6}{9}\right) dt = \frac{x^3}{3} + \frac{x^7}{63},$$

$$y_3(x) = \int_0^x [t^2 + y_2^2(t)]\,dt = \int_0^x \left(t^2 + \frac{t^6}{9} + \frac{2t^{10}}{3\cdot 63} + \frac{t^{14}}{63^2}\right) dt =$$

$$= \frac{x^3}{3} + \frac{x^7}{63} + \frac{2x^{11}}{2079} + \frac{x^{15}}{59535}.$$

The absolute error in the third approximation does not exceed the value

$$|y_3(x) - y(x)| \leqslant \frac{2}{3!}\left(\frac{1}{2}\right)^3 2^2 = \frac{1}{6},$$

here

$$N = \max_D \left|\frac{\partial f}{\partial y}\right| = \max_D |2y| = 2.$$

Remark. The function $f(x, y)$ must satisfy all the conditions of the existence and uniqueness theorem for the initial value problem.

The following example [3] shows that the continuity of the function $f(x, y)$ alone is not enough for the successive approximations to converge.

Let the function $f(x, y)$ be defined as follows:

$$f(x, y)=\begin{cases} 0 & \text{for } x=0,\ -\infty<y<+\infty, \\ 2x & \text{for } 0<x\leqslant 1,\ -\infty<y<0, \\ 2x-\frac{4y}{x} & \text{for } 0<x\leqslant 1,\ 0\leqslant y\leqslant x^2, \\ -2x & \text{for } 0<x\leqslant 1,\ x^2<y<+\infty. \end{cases}$$

In the set $0 \leqslant x \leqslant 1$, $-\infty < y < +\infty$ the function $f(x, y)$ is continuous and bounded by the constant $M = 2$. For the initial point $(x, y) = (0, 0)$ successive approximations with $0 \leqslant x \leqslant 1$ are of the form

$$y_0(x)=0,$$

$$y_1(x)=\int_0^x f(x, y_0(x))\,\mathrm{d}x=x^2,$$

$$y_2(x)=\int_0^x f(x, x^2)\,\mathrm{d}x=\int_0^x \left(2x-\frac{4x^2}{x}\right)\mathrm{d}x=-x^2$$

and in general

$$y_{2n-1}(x)=x^2, \quad y_{2n}(x)=-x^2, \quad n=1, 2, \ldots .$$

Therefore the sequence $\{y_n(x)\}$ has no limit for each $x \neq 0$, i.e. the successive approximations do not converge. Note also that neither of the converging sequences $\{y_{2n-1}(x)\}$ and $\{y_{2n}(x)\}$ converges to a solution, since

$$y'_{2n-1}(x)=2x\neq f(x, x^2)=-2x,$$
$$y'_{2n}(x)=-2x\neq f(x, -x^2)=2x.$$

If, however, the successive approximations converge, then the solution obtained may turn out to be nonunique, as is shown by the following example: $y' = y^{1/3}$.

Take the initial condition $y(0) = 0$; then

$$y(x)=\int_0^x y^{1/3}(t)\,\mathrm{d}t.$$

Taking the function $y_0(x) \equiv 0$ as the zero-order approximation we have

$$y_1(x) \equiv 0, \quad y_2(x) \equiv 0, \ \ldots, \quad y_n(x) \equiv 0, \ \ldots$$

so that all the successive approximations are zero and therefore they converge to a function which is identically zero. On the other hand, the function $y(x) = \left(\frac{2x}{3}\right)^{3/2}$ is also a solution of this problem existing on the half-line $x \geqslant 0$.

In the following problems find the first three successive approximations:

41. $y' = x^2 - y^2, \qquad y|_{x=-1} = 0.$
42. $y' = x + y^2, \qquad y_{x=0} = 0.$
43. $y' = x + y, \qquad y|_{x=0} = 1.$
44. $y' = 2y - 2x^2 - 3, \qquad y|_{x=0} = 2.$
45. $xy' = 2x - y, \qquad y|_{x=1} = 2.$

4. Equations with variables separable and equations reducible to them

A differential equation of the form $\varphi(y)\,dy = f(x)\,dx$ is called *an equation with separated variables.*

An equation of the form

$$\varphi_1(x)\,\psi_1(y)\,dx = \varphi_2(x)\,\psi_2(y)\,dy$$

in which the coefficients of the differentials are factors depending on x alone and on y alone is called *an equation with variables separable.*

Dividing by the product $\psi_1(y)\,\varphi_2(x)$ reduces it to an equation with separated variables:

$$\frac{\varphi_1(x)}{\varphi_2(x)}\,dx = \frac{\psi_2(y)}{\psi_1(y)}\,dy.$$

The general integral of this equation is of the form

$$\int \frac{\varphi_1(x)}{\varphi_2(x)}\,dx - \int \frac{\psi_2(y)}{\psi_1(y)}\,dy = C.$$

Remark. Dividing by $\psi_1(y)\,\varphi_2(x)$ may lead to the loss of particular solutions making the product $\psi_1(y)\,\varphi_2(x)$ zero. ◆

A differential equation of the form

$$\frac{dy}{dx}=f(ax+by+c),$$

where a, b, and c are constants, is converted into an equation with variables separable by the replacement of the variables $z = ax + by + c$.

Example 1. Solve the equation

$$3e^x \tan y\, dx + (2 - e^x) \sec^2 y\, dy = 0.$$

Solution. We divide both sides of the equation by the product $\tan y \times (2 - e^x)$:

$$\frac{3e^x\,dx}{2-e^x}+\frac{\sec^2 y\,dy}{\tan y}=0.$$

This is an equation with separated variables. Integrating it we find that

$$-3 \ln |2 - e^x| + \ln |\tan y| = C_1.$$

On taking the exponent of both sides we have

$$\frac{|\tan y|}{|2-e^x|^3}=e^{C_1} \quad \text{or} \quad \left|\frac{\tan y}{(2-e^x)^3}\right|=e^{C_1}$$

whence

$$\frac{\tan y}{(2-e^x)^3}=\pm e^{C_1}.$$

Denoting $\pm e^{C_1} = C$ we get

$$\frac{\tan y}{(2-e^x)^3}=C \quad \text{or} \quad \tan y - C(2-e^x)^3=0.$$

This is the general integral of the given equation.

It was assumed in dividing by the product $\tan y \times (2 - e^x)$ that neither of the factors vanishes. Equating each factor to zero gives respectively

$$y = k\pi \quad (k = 0, \pm 1, \pm 2, \ldots), \quad x = \ln 2.$$

Direct substitution in the original equation shows that $y = k\pi$ and $x = \ln 2$ are solutions of this equation. They can be obtained formally from the general integral when $C = 0$ and $C = \infty$. This means that the constant C is replaced by $1/C_2$, which makes the general integral assume the form

$$\tan y-\frac{1}{C_2}(2-e^x)^3=0 \quad \text{or} \quad C_2 \tan y-(2-e^x)^3=0.$$

Putting in the last equation $C_2 = 0$, which corresponds to $C = \infty$, we have $(2 - e^x)^3 = 0$, whence we obtain the solution $x = \ln 2$ of the original equation. So the functions $y = k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ and $x = \ln 2$ are particular solutions of the given equation. Therefore the final answer is

$$\tan y - C(2 - e^x)^3 = 0.$$

Example 2. Find a particular solution of the equation

$$(1 + e^x)\, yy' = e^x$$

satisfying the initial condition $y|_{x=0} = 1$.

Solution. We have

$$(1+e^x)\, y \frac{dy}{dx} = e^x.$$

Separating the variables we get

$$y\, dy = \frac{e^x\, dx}{1+e^x}.$$

Integrating we find the general integral

$$\frac{y^2}{2} = \ln(1+e^x) + C. \qquad (1)$$

Putting in (1) $x = 0$ and $y = 1$ we have

$$1/2 = \ln 2 + C \text{ whence } C = 1/2 - \ln 2.$$

Substituting the obtained value of C in (1) we get the particular solution

$$y^2 = 1 + \ln\left(\frac{1+e^x}{2}\right)^2, \text{ whence } y = \pm\sqrt{1+\ln\left(\frac{1+e^x}{2}\right)^2}.$$

It follows from the initial condition that $y > 0$ ($y|_{x=0} = 1 > 0$), therefore we use the positive sign before the radical. So the sought-for particular solution is

$$y = \sqrt{1 + \ln\left(\frac{1+e^x}{2}\right)^2}.$$

Example 3. Find particular solutions of the equation

$$y' \sin x = y \ln y$$

satisfying the initial conditions:

(a) $y|_{x=\pi/2} = e$; (b) $y = |_{x=\pi/2} = 1$.

Solution. We have

$$\frac{dy}{dx}\sin x = y\ln y.$$

We separate the variables

$$\frac{dy}{y\ln y}=\frac{dx}{\sin x}.$$

Integrating we find the general integral

$$\ln|\ln y| = \ln\left|\tan\frac{x}{2}\right| + \ln C.$$

On taking the exponent of both sides we get

$$\ln y = C\tan\frac{x}{2} \quad \text{or} \quad y = e^{C\tan\frac{x}{2}}$$

which is the general solution of the original equation.

(a) Put $x = \pi/2$, $y = e$, then $e = e^{C\tan\frac{\pi}{4}}$, whence $C = 1$. The required particular solution is $y = e^{\tan\frac{x}{2}}$;

(b) putting $x=\frac{\pi}{2}$, $y = 1$ in the general solution we have $1 = e^{C\tan\frac{\pi}{4}}$, whence $C = 0$. The required particular solution is $y \equiv 1$. ◆

Notice that in the process of obtaining the general solution the constant C was under the logarithm sign and hence $C = 0$ should be regarded as the limiting value. This particular solution $y = 1$ is contained among the zeros of the product $y\ln y\sin x$ by which we divided both sides of the given equation.

Example 4. Find the curve passing through the point $(0, -2)$ such that the slope of the tangent at any of its points is equal to the ordinate of that point increased by 3 units.

Solution. Starting from the visualization of the first derivative we obtain the differential equation of the family of curves satisfying the property required in the problem, namely

$$\frac{dy}{dx} = y + 3.$$

Separating the variables and integrating we obtain the general solution

$$y = Ce^{x} - 3. \tag{2}$$

Since the desired curve must pass through the point (0, —2), i.e. $y\,|_{x=0} = -2$, when $x = 0$ (2) yields $-2 = C - 3$, whence $C = 1$. The desired curve will be determined by the equation

$$y = e^{x} - 3.$$

Example 5. Find the curve possessing the property such that the length of its arc contained between some two points

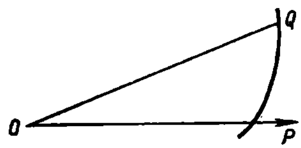

Fig. 11

P and Q is proportional to the difference between the distances of the points P and Q from the fixed point O.

Solution. If we fix the point P, then the arc QP will vary in direct proportion to the difference between OQ and the constant OP. We introduce polar coordinates, taking the point O as the pole and OP as the polar axis (Fig. 11). The differential of the arc of the curve in polar coordinates is

$$(ds)^2 = (dr)^2 + (r\,d\varphi)^2.$$

Hence for our problem we have

$$k^2 (dr)^2 = (dr)^2 + (r\,d\varphi)^2 \quad \text{or} \quad d\varphi = \sqrt{k^2 - 1}\,\frac{dr}{r} = \frac{1}{a}\,\frac{dr}{r}.$$

Integrating we find $r = C\,e^{a\varphi}$ (a logarithmic spiral).

Example 6. Let us assume that at a constant temperature the dissolution rate of a solid in a liquid is directly proportional to the amount of the substance still capable of dissolving in the liquid before it is saturated (it is supposed that the substances in the solution do not act chemically upon one another and that the solution is still far from being saturated, since otherwise the linear law for dissolution

rate cannot be applied). Find the time dependence of the amount of dissolved substance.

Solution. Let P be the amount of substance giving saturated solution and x be the amount of already dissolved substance. Then we obtain the differential equation

$$\frac{dx}{dt} = k(P-x),$$

where k is a proportionality coefficient known from the experiment and t is the time. Separating the variables we find

$$\frac{dx}{P-x} = k\,dt.$$

Integrating we get

$$\ln|x-P| = \ln C - kt, \quad \text{whence } x = P + Ce^{-kt}.$$

At the initial time $t=0$ we have $x=0$, therefore $C=-P$, so that finally

$$x = P(1-e^{-kt}).$$

Example 7. A cylindrical vessel of volume V_0 contains atmospheric air which is compressed adiabatically (without heat entering or leaving the system) to a volume V_1. Calculate the compression work.

Solution. It is known that an adiabatic process is characterized by the Poisson equation

$$p/p_0 = (V_0/V)^k, \tag{3}$$

where V_0 is the initial volume of gas, p_0 is the initial pressure of the gas, and k is the constant for the gas. Denote by V and p respectively the volume and pressure of the gas at the time the piston is at a height h, and by S the area of the piston. Then the lowering of the piston by a quantity dh will decrease the volume of the gas by the quantity $dV = S\,dh$, the work done being

$$dW = -p\,S\,dh \text{ or } dW = -p\,dV. \tag{4}$$

Finding p from (3) and substituting into (4) we obtain a differential equation of the process

$$dW = -\frac{p_0V_0^k}{V^k}\,dV.$$

Integrating the equation we have

$$W = -p_0V_0^k \int \frac{dV}{V^k} = \frac{p_0V_0^k}{(k-1)V^{k-1}} + C, \qquad k \neq 1. \tag{5}$$

According to the initial condition $W|_{V=V_0} = 0$, (5) yields

$$C = -p_0V_0/(k-1).$$

Thus the work of adiabatic compression (from V_0 to V) is

$$W = \frac{p_0V_0}{k-1}\left[\left(\frac{V_0}{V}\right)^{k-1} - 1\right].$$

When $V = V_1$ we get

$$W_1 = \frac{p_0V_0}{k-1}\left[\left(\frac{V_0}{V_1}\right)^{k-1} - 1\right].$$

Example 8. Find the solution of the equation

$$x^3 \sin y \cdot y' = 2 \tag{6}$$

satisfying the condition

$$y \to \frac{\pi}{2} \quad \text{as} \quad x \to \infty. \tag{7}$$

Solution. Separating the variables and integrating we find the general integral of equation (6)

$$\cos y = \frac{1}{x^2} + C.$$

Condition (7) gives $\cos\frac{\pi}{2} = C$, i.e. $C = 0$, so that the particular integral will be of the form $\cos y = 1/x^2$. There are an infinite number of particular solutions of the form

$$y = \pm \arccos \frac{1}{x^2} + 2\pi n, \quad n = 0, \ \pm 1, \ \pm 2, \ \ldots \tag{8}$$

corresponding to it. Among these solutions there is only one satisfying condition (7). This solution will be found proceeding to the limit as $x \to \infty$ in equation (8):

$$\frac{\pi}{2} = \pm \arccos 0 + 2\pi n \quad \text{or} \quad \frac{\pi}{2} = \pm \frac{\pi}{2} + 2\pi n$$

whence

$$\frac{1}{2} = \pm \frac{1}{2} + 2n. \tag{9}$$

It can easily be seen that equation (9) has two solutions, namely, $n = 0$ and $n = 1/2$, the solution $n = 1/2$, corre-

sponding to the negative sign before $\arccos\frac{1}{x^2}$, being unsuitable (n must be an integer or zero). Thus the required particular solution of equation (6) is

$$y = \arccos\frac{1}{x^2}.$$

Integrate the following equations:

46. $(1+y^2)\,dx + (1+x^2)\,dy = 0.$

47. $(1+y^2)\,dx + xy\,dy = 0.$

48. $y'\sin x - y\cos x = 0, \qquad y\big|_{x=\frac{\pi}{2}} = 1.$

49. $(1+y^2)\,dx = x\,dy.$

50. $x\sqrt{1+y^2} + yy'\sqrt{1+x^2} = 0.$

51. $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0, \quad y\big|_{x=0} = 1.$

52. $e^{-y}(1+y') = 1.$

53. $y\ln y\,dx + x\,dy = 0, \qquad y\big|_{x=1} = 1.$

54. $y' = a^{x+y}\ (a>0,\ a\neq 1).$

55. $e^y(1+x^2)\,dy - 2x(1+e^y)\,dx = 0.$

56. $2x\sqrt{1-y^2} = y'(1+x^2).$

57. $e^x\sin^3 y + (1+e^{2x})\cos y\cdot y' = 0.$

58. $y^2\sin x\,dx + \cos^2 x\ln y\,dy = 0.$

59. $y' = \sin(x-y).$

60. $y' = ax + by + c$ (a, b, c—const).

61. $(x+y)^2 y' = a^2.$

62. $y + xy' = a(1+xy), \qquad y\big|_{x=\frac{1}{a}} = -a.$

63. $(a^2+y^2)\,dx + 2x\sqrt{ax-x^2}\,dy = 0, \qquad y_{x=a} = 0.$

64. $y' = \sin(x-y) = \sin(x+y), \qquad y\big|_{x=\pi} = \frac{\pi}{2}.$

65. Find the curve passing through the point (0, —2) such that the slope of the tangent at any of its points is

equal to the ordinate of that point augmented by a factor of 3.

66. Find a curve for which an area Q bounded by the curve, the Ox axis and two ordinates, $X = 0$, $X = x$, is a given function of y : $Q = a^2 \ln (y/a)$.

67. A material point with a mass of 1 g is moving in a straight line under a force directly proportional to the time counted from the time $t = 0$ and inversely proportional to the velocity of travel of the point. At the time $t = 10$ sec the velocity was 50 cm/sec and the force was 4 dynes. What will the velocity be in one minute after the start of the motion?

68. Prove that a curve possessing the property that all its normals pass through a fixed point is a circle.

69. A bullet enters a board of thickness $h = 10$ cm with a velocity $v_0 = 200$ m/sec and goes out with a velocity $v_1 = 80$ m/sec. Considering that the resistance force of the board to the movement of the bullet is proportional to the square of the velocity of the bullet, find the time required for the bullet to pass through the board.

70. A ship slows down due to the resistance of the water which is proportional to the velocity of the ship. The initial velocity of the ship is 10 m/sec, its velocity in 5 sec becoming 8 m/sec. When will the velocity decrease to 1 m/sec?

71. Prove that a curve such that at any point the slope of its tangent is proportional to the abscissa of the point of tangency is a parabola.

72. By Newton's law the cooling rate of a body in the air is proportional to the difference between the temperature T of the body and the air temperature T_0. If the air temperature is 20°C and in 20 min the body cools from 100° to 60°, in what time will its temperature drop to 30°?

73. Find a curve such that the slope of the tangent at some point is n times that of the straight line joining that same point and the origin of coordinates.

74. Determine the distance S moved by a body in time t, if its velocity is proportional to the distance travelled and if the body moves over a distance of 100 m in 10 sec and 200 m in 15 sec.

75. The bottom of a tank with a capacity of 300 l is covered with salt. Assuming that the dissolution rate of salt is proportional to the difference between the concen-

tration at a given time and that of the saturated solution (1 kg of salt per 3 l of water) and that the given amount of pure water dissolves 1/3 kg of salt in one minute, find the amount of salt in the solution at the end of 1 hour.

76. A certain amount of insoluble substance contains in its pores 10 kg of salt. Subjecting it to the action of 90 l of water it was found that half the salt it contained had dissolved at the end of 1 hour. How much salt would have dissolved in the same time, if the amount of water had been doubled? The dissolution rate is proportional to the amount of undissolved salt and the difference between the concentration of solution at a given time and that of the saturated solution (1 kg per 3 l).

77. Find a curve possessing the property that a segment of a tangent to the curve contained between the coordinate axes is bisected at the point of tangency.

78. A certain amount of substance containing 3 kg of moisture was placed into a room with a capacity of 100 cu. m and an initial air humidity of 25%. At the same temperature the saturated air contains 0.12 kg of moisture per 1 cu. m. If in twenty-four hours the substance lost half of its moisture, how much moisture would remain in it at the end of another twenty-four hours?

Hint. Moisture contained in a porous substance evaporates into the environment at a rate proportional to the amount of moisture in the substance as well as to the difference between the humidity of the ambient air and that of the saturated air.

79. A certain amount of insoluble substance containing in its pores 2 kg of salt is subjected to the action of 30 l of water. 1 kg of the salt is dissolved in 5 min. In what time will 99% of the initial amount of salt be dissolved?

. 80. A brick wall is 30 cm thick. Find the dependence of temperature on the distance of a point from the outer edge of the wall if the temperature is 20 °C at the inner surface of the wall and 0°C at its outer surface. Find also the amount of heat the wall gives off (per 1 sq. cm) in twenty-four hours.

Hint. By virtue of Newton's law the velocity Q with which heat spreads over an area element A perpendicular to the Ox axis is $Q = -kS\frac{dT}{dt}$, where k is the thermal conductivity coefficient of the substance ($k = 0.0015$), T is the temperature, t is the time, and S is the area of A.

81. Show that the equation $\frac{dy}{dx}=\frac{y}{x}$ subject to the initial condition $y|_{x=0}=0$ has an infinite number of solutions of the form $y=Cx$. The same equation subject to the initial condition $y|_{x=0}=y_0\neq 0$ has no solutions. Construct the integral curves.

82. Show that the problem $\frac{dy}{dx}=y^\alpha$, $y\big|_{x=0}=0$ has at least two solutions for $0<\alpha<1$ and one solution for $\alpha=1$. Construct the integral curves for $\alpha=1/2,\ 1$.

83. Find the solution of the equation $\frac{dy}{dx}=y|\ln y|^\alpha$, $\alpha>0$ satisfying the initial condition $y|_{x=0}=0$. For what values of α has the problem a unique solution?

84. Show that the tangents to all integral curves of the differential equation $y'+y\tan x=x\tan x+1$ at the points of intersection with the Oy axis are parallel. Determine the angle at which the integral curves cut the Oy axis.

Integrate the following differential equations:

85. $\cos y'=0$.

86. $e^{y'}=1$.

87. $\sin y'=x$.

88. $\ln y'=x$.

89. $\tan y'=0$.

90. $e^{y'}=x$.

91. $\tan y'=x$.

In the following problems find the solutions of the equations satisfying the indicated conditions:

92. $x^2y'\cos y+1=0,\quad y\to\frac{16}{3}\pi,\quad x\to+\infty.$

93. $x^2y'+\cos 2y=1,\quad y\to\frac{10}{3}\pi,\quad x\to+\infty.$

94. $x^3y'-\sin y=1,\quad y\to 5\pi,\quad x\to\infty.$

95. $(1+x^2)\,y'-\frac{1}{2}\cos^2 2y=0,\quad y\to\frac{7}{2}\pi,\quad x\to-\infty.$

96. $e^y=e^{4y}y'+1$, y being bounded when $x\to+\infty$.

97. $(x+1)\,y'=y-1$, y being bounded when $x\to+\infty$.

98. $y'=2x(\pi+y)$, y being bounded when $x\to\infty$.

99. $x^2y'+\sin 2y=1,\quad y\to\frac{11}{4}\pi,\quad x\to+\infty.$

5. Homogeneous equations and equations reducible to them

5.1. Homogeneous equations. A function $f(x, y)$ is said to be a *homogeneous function* of its variables of degree n if the identity $f(tx, ty) \equiv t^n f(x, y)$ is valid.

For instance, the function $f(x, y) = x^2 + y^2 - xy$ is a homogeneous function of the second degree, since

$$f(tx, ty) = (tx)^2 + (ty)^2 - (tx)(ty) = t^2(x^2 + y^2 - xy)$$
$$= t^2 f(x, y).$$

For $n = 0$ we have the function of zero degree. For instance, $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ is a function of zero degree since

$$f(tx, ty) = \frac{(tx)^2 - (ty)^2}{(tx)^2 + (ty)^2} = \frac{t^2(x^2 - y^2)}{t^2(x^2 + y^2)} = \frac{x^2 - y^2}{x^2 + y^2} = f(x, y).$$

A differential equation of the form $\frac{dy}{dx} = f(x, y)$ is said to be *homogeneous in* x and y if $f(x, y)$ is a homogeneous function of its variables of zero degree. A homogeneous equation can always be represented by

$$\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right). \tag{1}$$

Introducing a new desired function $u = y/x$ equation (1) can be reduced to an equation with variables separable:

$$x\frac{du}{dx} = \varphi(u) - u.$$

If $u = u_0$ is the solution of the equation $\varphi(u) - u = 0$, then the solution of the homogeneous equation will be $u = u_0$ or $y = u_0 x$ (a straight line through the origin of coordinates).

Remark. When solving homogeneous equations it is not obligatory to reduce them to the form (1). It is possible to make a direct substitution $y = ux$.

Example 1. Solve the equation $xy' = \sqrt{x^2 - y^2} + y$.

Solution. Write the equation in the following form

$$y' = \sqrt{1 - \left(\frac{y}{x}\right)^2} + \frac{y}{x},$$

which makes it homogeneous in x and y. Put $u = y/x$ or $y = ux$. Then $y' = xu' = u$. Substituting the expressions

for y and y' in the equation we obtain

$$x\frac{du}{dx}=\sqrt{1-u^2}.$$

We separate the variables:

$$\frac{du}{\sqrt{1-u^2}}=\frac{dx}{x}.$$

Hence we find by integration

$$\arcsin u = \ln|x| + \ln C_1 \quad (C_1>0) \text{ or } \arcsin u = \ln C_1|x|.$$

Since $C_1|x| = \pm C_1 x$, then denoting $\pm C_1 = C$ we have $\arcsin u = \ln Cx$, where $|\ln Cx| \leqslant \pi/2$ or $e^{-\pi/2} \leqslant Cx \leqslant e^{\pi/2}$. Replacing u by y/x we obtain the general integral

$$\arcsin\frac{y}{x}=\ln Cx.$$

Hence the general solution is $y = x \sin \ln Cx$.

When separating the variables we divided both sides of the equation by the product $x\sqrt{1-u^2}$, we might therefore lose the solutions making this product zero.

Now put $x = 0$ and $\sqrt{1-u^2} = 0$. But $x \neq 0$ by virtue of the substitution $u = y/x$ and from the second equation we find that $1-\frac{y^2}{x^2}=0$, whence $y = \pm x$. A direct check shows that the functions $y = -x$ and $y = x$ are also solutions of the given equation.

Example 2. Consider a family of integral curves C_α of the homogeneous equation

$$y' = \varphi(y/x). \tag{2}$$

Show that the tangents to the curves determined by the homogeneous differential equation (2) are parallel at corresponding points*.

Solution. By the definition of corresponding points we have $\frac{y}{x}=\frac{y_1}{x_1}$, so that by virtue of equation (2) itself

$$y' = y_1',$$

* We shall refer to those points on curves C_α as *corresponding* which lie on one ray issuing from the origin of coordinates.

where y' and y'_1 are the slopes of the tangents to the integral curves C_α and C_{α_1} at the points M and M_1 respectively (Fig. 12).

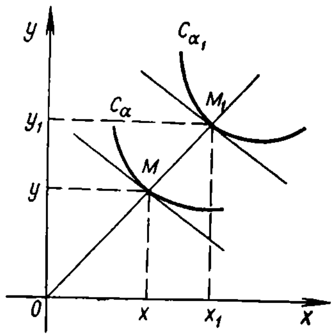

Fig. 12

Integrate the following equations:

100. $xy' = y + x\cos^2\frac{y}{x}$.

101. $(x-y)\,dx + x\,dy = 0$.

102. $xy' = y(\ln y - \ln x)$.

103. $x^2\,dy = (y^2 - xy + x^2)\,dx$.

104. $xy' = y + \sqrt{y^2 - x^2}$.

105. $2x^2y' = x^2 + y^2$.

106. $(4x - 3y)\,dx + (2y - 3x)\,dy = 0$.

107. $(y - x)\,dx + (y + x)\,dy = 0$.

5.2. Equations reducible to homogeneous equations.

A. Consider a differential equation of the form

$$\frac{dy}{dx} = f\left(\frac{ax+by+c}{a_1x+b_1y+c_1}\right), \tag{3}$$

where a, b, c, a_1, b_1, and c_1 are constants and $f(u)$ is a continuous function of its variable u.

If $c = c_1 = 0$, then equation (3) is homogeneous and integrated as shown above.

If at least one of the numbers c, c_1 is nonzero, then two cases should be distinguished.

(1) The determinant $\Delta = \begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} \neq 0$. Introducing new variables ξ and η by means of the formulas $x = \xi + h$ and $y = \eta + k$, where h and k are yet undetermined constants, we rewrite equation (3) in the following form

$$\frac{d\eta}{d\xi} = f\left(\frac{a\xi + b\eta + ah + bk + c}{a_1\xi + b_1\eta + a_1h + b_1k + c_1}\right).$$

Taking h and k as a solution of the system of linear equations

$$\begin{cases} ah + bk + c = 0, \\ a_1h + b_1k + c_1 = 0 \end{cases} \quad (\Delta \neq 0) \tag{4}$$

we obtain the homogeneous equation

$$\frac{d\eta}{d\xi} = f\left(\frac{a\xi + b\eta}{a_1\xi + b_1\eta}\right).$$

On finding its general integral and replacing ξ by $x - h$ and η by $y - k$ we obtain the general integral of equation (3).

(2) The determinant $\Delta = \begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = 0$. The system (4) has no solutions in the general case and the method described above cannot be applied; in this case $\frac{a_1}{a} = \frac{b_1}{b} = \lambda$ and hence equation (2) has the form

$$\frac{dy}{dx} = f\left(\frac{ax + by + c}{\lambda(ax + by) + c_1}\right).$$

Substituting $z = ax + by$ reduces it to an equation with variables separable.

Example 3. Solve the equation

$$(x + y - 2)\,dx + (x - y + 4)\,dy = 0. \tag{5}$$

Solution. Consider the following system of linear algebraic equations

$$\begin{cases} x + y - 2 = 0, \\ x - y + 4 = 0. \end{cases}$$

The determinant of this system is

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0.$$

The system has a unique solution $x_0 = -1$, $y_0 = 3$. We substitute $x = \xi - 1$, $y = \eta + 3$. Then equation (5) takes the form

$$(\xi + \eta)\, d\xi + (\xi - \eta)\, d\eta = 0. \tag{6}$$

Equation (6) is homogeneous. Putting $\eta = u\xi$ we get

$$(\xi + \xi u)\, d\xi + (\xi - \xi u)(\xi\, du + u\, d\xi) = 0,$$

whence

$$(1 + 2u - u^2)\, d\xi + \xi (1 - u)\, du = 0.$$

We separate the variables

$$\frac{d\xi}{\xi} + \frac{1-u}{1+2u-u^2}\, du = 0.$$

By integrating we find that

$$\ln|\xi| + \frac{1}{2}\ln|1 + 2u - u^2| = \ln C \text{ or } \xi^2 (1 + 2u - u^2) = C.$$

We return to the variables x, y:

$$(x+1)^2 \left[1 + 2\frac{y-3}{x+1} - \frac{(y-3)^2}{(x+1)^2}\right] = C_1$$

or

$$x^2 + 2xy - y^2 - 4x + 8y = C \ (C = C_1 + 14).$$

Example 4. Solve the equation $(x + y + 1)\, dx + (2x + 2y - 1)\, dy = 0$.

Solution. The system of linear algebraic equations

$$\begin{cases} x + y + 1 = 0, \\ 2x + 2y - 1 = 0 \end{cases}$$

is incompatible. In this case the method used in the previous example is not suitable. To integrate the equation we use the substitution $x + y = z$, $dy = dz - dx$. The equation takes the form

$$(2 - z)\, dx + (2z - 1)\, dz = 0.$$

Separating the variables we get

$$dx - \frac{2z-1}{z-2}\, dz = 0, \text{ hence } x - 2z - 3\ln|z - 2| = C.$$

Returning to the variables x, y we obtain the general integral of the given equation $x + 2y + 3\ln|x + y - 2| = C$.

Solve the following equations:

108. $x + y - 2 + (1 - x)\, y' = 0.$

109. $(3y - 7x + 7)\, dx - (3x - 7y - 3)\, dy = 0.$

110. $(x + y - 2)\, dx + (x - y + 4)\, dy = 0.$

111. $(x + y)\, dx + (x - y - 2)\, dy = 0.$

112. $2x + 3y - 5 + (3x + 2y - 5)\, y' = 0.$

113. $8x + 4y + 1 + (4x + 2y + 1)\, y' = 0.$

114. $(x - 2y - 1)\, dx + (3x - 6y + 2)\, dy = 0.$

115. $(x + y)\, dx + (x + y - 1)\, dy = 0.$

B. Sometimes an equation can be reduced to a homogeneous one by substituting z^{α} for y. This is the case when all the terms in the equation are of the same degree once the variable x is assigned degree 1, the variable y degree α and the derivative $\frac{dy}{dx}$ degree $\alpha - 1$.

Example 5. Solve the equation

$$(x^2y^2 - 1)\, dy + 2xy^3\, dx = 0. \tag{7}$$

Solution. We replace y by z^{α}, dy by $\alpha z^{\alpha-1}\, dz$, where α is as yet an arbitrary number to be chosen later on. Substituting in the equation the expressions for y and dy we get

$$(x^2z^{2\alpha} - 1)\, \alpha z^{\alpha-1}\, dz + 2xz^{3\alpha}\, dx = 0$$

or

$$(x^2z^{3\alpha-1} - z^{\alpha-1})\, \alpha\, dz + 2xz^{3\alpha}\, dx = 0.$$

Notice that $x^2z^{3\alpha-1}$ is of degree $2 + 3\alpha - 1 = 3\alpha + 1$, $z^{\alpha-1}$ is of degree $\alpha - 1$, and $xz^{3\alpha}$ is of degree $1 + 3\alpha$. The equation obtained will be homogeneous if all the terms are of the same degree, i.e. if the condition $3\alpha + 1 = \alpha - 1$ or $\alpha = -1$ is fulfilled.

Put $y = 1/z$; the original equation will take the form

$$\left(\frac{1}{z^2} - \frac{x^2}{z^4}\right) dz + 2\,\frac{x}{z^3}\, dx = 0$$

or

$$(z^2 - x^2)\, dz + 2zx\, dx = 0.$$

Now put $z = ux$, $dz = u\, dx + x\, du$. Then this equation will assume the form $(u^2 - 1)\,(u\, dx + x\, du) + 2u\, dx = 0$,

whence

$$u(u^2+1)\,dx + x(u^2-1)\,du = 0.$$

We separate the variables in the equation:

$$\frac{dx}{x} + \frac{u^2-1}{u^3+u}\,du = 0.$$

Integrating we find

$$\ln|x| + \ln(u^2+1) - \ln|u| = \ln C \quad\text{or}\quad \frac{x(u^2+1)}{u} = C.$$

Replacing u by $1/xy$ we obtain the general integral of the given equation:

$$1 + x^2y^2 = Cy.$$

Equation (7) has another obvious solution $y = 0$ which is obtained from the general integral when $C \to \infty$, if we write the integral as $y = (1 + x^2y^2)/C$ and then proceed to the limit when $C \to \infty$. Thus the function $y = 0$ is a particular solution of the original equation.

Integrate the following equations:

116. $2xy'(x - y^2) + y^3 = 0.$

117. $4y^6 + x^3 = 6xy^5y'.$

118. $y\left(1 + \sqrt{x^2y^4+1}\right)dx + 2x\,dy = 0.$

119. $(x + y^3)\,dx + 3(y^3 - x)\,y^2\,dy = 0.$

120. Find a curve possessing the property that the length of the perpendicular dropped from the origin of coordinates to a tangent is equal to the abscissa of the point of tangency.

121. Determine a curve such that the ratio between the length of a segment intercepted by a tangent on the Oy axis and the length of the radius vector is constant.

122. Using rectangular coordinates find the shape of a mirror reflecting in a way parallel to a given direction all the rays issuing from a given point.

123. Find a curve for which the length of the segment intercepted on the ordinate axis by a normal drawn at some point of the curve is equal to the distance of that point from the origin.

124. Find a curve for which the product of the abscissa of some point and the length of the segment intercepted by a normal on the Oy axis is twice the square of the distance from that point to the origin.

6. Linear equations of the first order. The Bernoulli equation

6.1. **Linear equations of the first order.** *A linear differential equation of the first order* is an equation linear in an unknown function and its derivative. It is of the form

$$\frac{dy}{dx}+p(x)\,y=q(x) \tag{1}$$

where $p(x)$ and $q(x)$ are given functions of x continuous in the range in which it is required to integrate equation (1).

If $q(x)\equiv 0$, then equation (1) is called *a homogeneous linear equation.* It is an equation with variables separable and has the general solution

$$y=C\,e^{-\int p(x)dx}.$$

The general solution of a nonhomogeneous equation can be found by *the method of variation of an arbitrary constant* which consists in finding the solution of equation (1) in the form

$$y=C(x)\,e^{-\int p(x)dx},$$

where $C(x)$ is a new unknown function of x.

Example 1. Solve the equation

$$y'+2xy=2x\,e^{-x^2}. \tag{2}$$

Solution. Use the method of variation of a constant. Consider the homogeneous equation

$$y'+2xy=0$$

corresponding to the given nonhomogeneous equation. It is an equation with variables separable. Its general solution is of the form

$$y=C\,e^{-x^2}.$$

The general solution of the nonhomogeneous equation is sought in the form

$$y=C(x)\,e^{-x^2}, \tag{3}$$

where $C(x)$ is an unknown function of x. Substituting (3) in (2) we obtain $C(x)=2x$, whence $C(x)=x^2+C$. So

the general solution of the nonhomogeneous equation is

$$y = (x^2 + C)\, e^{-x^2},$$

where C is the integration constant.

Remark. It may turn out that the differential equation is linear in x as a function of y. The normal form of such an equation is

$$\frac{dx}{dy} + r(y)\, x = \varphi(y).$$

Example 2. Solve the equation $\frac{dy}{dx} = \frac{1}{x\cos y + \sin 2y}$.

Solution. This equation is linear if one considers x as a function of y:

$$\frac{dx}{dy} - x\cos y = \sin 2y. \tag{4}$$

We use the method of variation of an arbitrary constant. We first solve the corresponding homogeneous equation:

$$\frac{dx}{dy} - x\cos y = 0,$$

which is an equation with variables separable. Its general solution is of the form

$$x = C\, e^{\sin y}, \quad C = \text{const}.$$

The general solution of equation (4) is sought in the form

$$x = C(y)\, e^{\sin y}, \tag{5}$$

where $C(y)$ is an unknown function of y. Substituting (5) into (4) we get

$$C'(y)\, a^{\sin y} + C(y)\, e^{\sin y}\cos y - C(y)\, e^{\sin y}\cos y = \sin 2y$$

or

$$C'(y) = e^{-\sin y}\sin 2y.$$

Hence, integrating by parts, we have

$$C(y) = \int e^{-\sin y}\sin 2y\, dy = 2\int e^{-\sin y}\cos y \sin y\, dy$$

$$= 2\int \sin y\, d(-e^{-\sin y}) = 2\,(-\sin y\, e^{-\sin y}$$

$$+ \int e^{-\sin y}\cos y\, dy) = 2\,(-\sin y\, e^{-\sin y} - e^{-\sin y}) + \ ;$$

so

$$C(y) = -2e^{-\sin y}(1 + \sin y) + C. \qquad (6)$$

Substituting (6) in (5) we obtain the general solution of equation (4) and hence that of the given equation:

$$x = C\, e^{\sin y} - 2(1 + \sin y). \ \blacklozenge$$

Equation (1) can be integrated in the following way as well. We put

$$y = u(x)\, v(x), \qquad (7)$$

where $u(x)$ and $v(x)$ are unknown functions of x one of which, for instance $v(x)$, may be chosen arbitrarily. Substituting (7) in (1) we get, after transformation,

$$vu' + (pv + v')\, u = q(x). \qquad (8)$$

Determining $v(x)$ from the condition $v' + pv = 0$ we then find from (8) the function $u(x)$ and hence the solution $y = uv$ of equation (1). It is possible to take as $v(x)$ any particular solution of the equation $v' + pv = 0$, $v \not\equiv 0$.

Example 3. Solve an initial value problem:

$$x(x-1)\, y' + y = x^2 (2x - 1), \qquad (9)$$

$$y\,|_{x=2} = 4. \qquad (10)$$

Solution. We seek the general solution of equation (9) in the form

$$y = u(x)\, v(x);$$

we have $y' = u'v + uv'$.

Substituting the expressions for y and y' into (9) we have

$$x(x-1)(u'v + uv') + uv = x^2 (2x - 1)$$

or

$$x(x-1)\, vu' + [x(x-1)\, v' + v]\, u = x^2 (2x - 1). \qquad (11)$$

The function $v = v(x)$ is found from the condition $x(x-1)\, v' + v = 0$. Taking any particular solution of the last equation, for instance $v = \frac{x}{x-1}$ and substituting it into (11) we obtain the equation $u' = 2x - 1$ from which we find the function $u(x)$; $u(x) = x^2 - x + C$. Therefore the general solution of equation (9) is

$$y = uv = (x^2 - x + C)\frac{x}{x-1} \quad \text{or} \quad y = \frac{Cx}{x-1} + x^2.$$

Using the initial condition (10) we obtain the equation $4 = C\frac{2}{2-1} + 2^2$, whence it follows that $C = 0$; so the solution of the initial value problem set above is

$$y = x^2.$$

Example 4. It is known that between current intensity i and electromotive force E in a circuit having resistance R and self-induction L there is a relation $E = Ri + L\frac{di}{dt}$, where R and L are constants. If E is regarded as a function of time t, then a nonhomogeneous linear equation for current intensity i is obtained:

$$\frac{di(t)}{dt} + \frac{R}{L}i(t) = \frac{E(t)}{L}.$$

Find current intensity $i(t)$ for the case where $E = E_0 = \text{const}$ and $i(0) = I_0$.

Solution. We have

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E_0}{L}, \tag{12}$$

$$i(0) = I_0. \tag{13}$$

The general solution of equation (12) is of the form

$$i(t) = \frac{E_0}{R} + Ce^{-\frac{R}{L}t}. \tag{14}$$

From this, using the initial condition (13), we get $C = I_0 - \frac{E_0}{R}$, so that the desired solution is

$$i(t) = \frac{E_0}{R} + \left(I_0 - \frac{E_0}{R}\right)e^{-\frac{R}{L}t}.$$

Hence one can see that as time t increases current intensity $i(t)$ approaches a constant value E_0/R.

Example 5. A family C_α of integral curves of the nonhomogeneous linear equation $y' + p(x)y = q(x)$ is given.

Show that the tangents at corresponding points* to the curves C_α determined by the linear equation intersect at one point (Fig. 13).

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* *Corresponding points* of curves C_α are such points that lie on the same straight line parallel to the ordinate axis.

Solution. Consider the tangent to some curve C_α at a point $M(x, y)$. The equation of the tangent at a point $M(x, y)$ is of the form

$$\eta - q(x)(\xi - x) = y[1 - p(x)(\xi - x)],$$

where ξ and η are moving coordinates of a tangency point. By the definition, at corresponding points x is constant

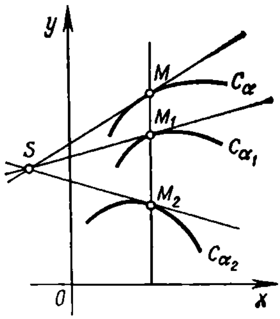

Fig. 13

and y is variable. Taking any two tangents to curves C_α at corresponding points we get for the coordinates of the point S of their intersection

$$\xi = x + \frac{1}{p(x)}, \quad \eta = + \frac{q(x)}{p(x)}. \tag{15}$$

Hence it is seen that all the tangents to the curves C_α at corresponding points (x being fixed) intersect at the same point

$$S\left(x + \frac{1}{p(x)}, \ + \frac{q(x)}{p(x)}\right).$$

Eliminating in the system (15) the variable x we obtain the equation of the locus of points S: $f(\xi, \eta) = 0$.

Example 6. Find a solution of the equation $y' - y = \cos x - \sin x$ satisfying the condition that y should be bounded when $x \to +\infty$.

Solution. The general solution of the equation is

$$y = C e^x + \sin x.$$

Any solution of the equation obtained from the general solution when $C \neq 0$ will be unbounded since when $x \to +\infty$

the function $\sin x$ is bounded and $e^x \to +\infty$. It follows that the given equation has a unique solution $y = \sin x$, bounded when $x \to +\infty$, which is obtained from the general solution when $C = 0$.

Solve the following linear equations. Solve, where indicated, the initial value problem:

125. $y' + 2y = e^{-x}$.

126. $x^2 = xy' = y, \quad y|_{x=1} = 0$.

127. $y' - 2xy = 2xe^{x^2}$.

128. $y' + 2xy = e^{-x^2}$.

129. $y' \cos x - y \sin x = 2x, \quad y|_{x=0} = 0$.

130. $xy' - 2y = x^3 \cos x$.

131. $y' - y \tan x = \dfrac{1}{\cos^3 x}, \quad y|_{x=0} = 0$.

132. $y'x \ln x - y = 3x^3 \ln^2 x$.

133. $(2x - y^2)\, y' = 2y$.

134. $y' + y \cos x = \cos x, \quad y|_{x=0} = 1$.

135. $y' = \dfrac{y}{2y \ln y + y - x}$.

136. $\left(e^{-\frac{y^2}{2}} - xy\right) dy - dx = 0$.

137. $y' - ye^x = 2xe^{e^x}$.

138. $y' + xe^x y = e^{(1-x)e^x}$.

139. Find current intensity $i(t)$ if $E(t) = E_0 \sin 2\pi nt$, $i(0) = I_0$, where $E_0, I_0 = \text{const}$.

140. A capacitor of capacitance Q is connected to a circuit with voltage E and resistance R. Determine the charge q of the capacitor at time t after connection.

141. A point of mass m moves in a straight line. It is acted upon by a force proportional to the time (the proportionality coefficient being k_1). In addition it overcomes the resistance of the medium proportional to its velocity (the proportionality coefficient being k_2). Find the time dependence of the velocity considering that at the starting point the velocity was zero.

142. Find the curves possessing the property that the intercept the tangent at any point of a curve cuts off on

the Oy axis is equal to the square of the abscissa of the tangency point.

143. Find a curve such that the intercept a tangent cuts off on the ordinate axis is half the sum of the coordinates of the tangency point.

144. Find the general solution of the nonhomogeneous linear equation of the first order $y' + p(x)y = q(x)$ if one particular solution, $y_1(x)$, is known.

145. Find the general solution of the first order nonhomogeneous linear equation $y' + p(x)y = q(x)$ if two particular solutions of it, $y_1(x)$ and $y_2(x)$, are known.

146. Show that a linear equation remains linear whatever replacements of the independent variable $x = \varphi(t)$, where $\varphi(t)$ is a differentiable function, are made.

147. Show that a linear equation remains linear whatever linear transformations of the sought-for function $y = \alpha(x)z + \beta(x)$, where $\alpha(x)$ and $\beta(x)$ are arbitrary differentiable functions, with $\alpha(x) \neq 0$ in the interval under consideration, take place.

In the problems below find solutions of the equations satisfying the indicated conditions:

148. $y' - y\ln 2 = 2^{\sin x}(\cos x - 1)\ln 2$, y being bounded when $x \to +\infty$.

149. $y' - y = -2e^{-x}$, $y \to 0$ as $x \to +\infty$.

150. $y'\sin x - y\cos x = -\dfrac{\sin^2 x}{x^2}$, $y \to 0$ as $x \to \infty$.

151. $x^2 y'\cos\dfrac{1}{x} - y\sin\dfrac{1}{x} = -1$, $y \to 1$ as $x \to \infty$.

152. $2xy' - y = 1 - \dfrac{2}{\sqrt{x}}$, $y \to -1$ as $x \to +\infty$.

153. $x^2 y' + y = (x^2 + 1)e^x$, $y \to 1$ as $x \to -\infty$.

154. $xy' + y = 2x$, y being bounded when $x \to 0$.

155. $y'\sin x + y\cos x = 1$, y being bounded when $\to 0$.

156. $y'\cos x - y\sin x = -\sin 2x$, $y \to 0$ as $x \to \pi/2$.

6.2. The Bernoulli equation. The Bernoulli equation is of the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n,$$

where $n \neq 0, 1$ (for $n = 0$ and $n = 1$ this equation is linear).

By the substitution $z = \frac{1}{y^{n-1}}$ the Bernoulli equation may be reduced to a linear form and integrated as a linear equation.

Example 7. Solve the Bernoulli equation $y' - xy = -xy^3$.

Solution. We divide both sides of the equation by y^3:

$$\frac{y'}{y^3} - x\frac{1}{y^2} = -x.$$

We use the replacement of the variable $\frac{1}{y^2} = z$, $-\frac{2y'}{y^3} = z'$, whence $\frac{y'}{y^3} = -\frac{1}{2}z'$. Upon substitution the last equation turns into the linear equation

$$-\frac{1}{2}z' - xz = -x \quad \text{or} \quad z' + 2xz = 2x$$

whose general solution is

$$z = 1 + c^{-x^2}.$$

Hence we obtain the general integral of the given equation

$$\frac{1}{y^2} = 1 + Ce^{-x^2} \quad \text{or} \quad y^2(1 + Ce^{-x}) = 1.$$

Remark. A Bernoulli equation can also be integrated by the method of variation of a constant, like a linear equation, and by the substitution $y(x) = u(x)\,v(x)$.

Example 8. Solve the Bernoulli equation

$$xy' + y = y^2 \ln x. \tag{16}$$

Solution. Let us use the method of variation of an arbitrary constant. The general solution of the corresponding homogeneous equation $xy' + y = 0$ is of the form $y = C/x$. We seek the general solution of equation (16) in the form

$$y = C(x)/x, \tag{17}$$

where $C(x)$ is a new unknown function.

Substituting (17) into (16) we have

$$C'(x) = C^2(x)\frac{\ln x}{x^2}.$$

To find $C(x)$ we have obtained an equation with variables separable from which, separating the variables and inte-

grating, we get

$$\frac{1}{C(x)}=\frac{\ln x}{x}+\frac{1}{x}+C;\quad C(x)=\frac{x}{1+Cx+\ln x}.$$

Thus the general solution of equation (16) is

$$y=\frac{1}{1+Cx+\ln x}. \quad \blacklozenge$$

A well found substitution of variables may help to reduce some nonlinear equations of the first order to linear equations or to Bernoulli equations.

Example 9. Solve the equation $y' + \sin y + x\cos y + x = 0$.

Solution. We write the equation in the form

$$y' + 2\sin\frac{y}{2}\cos\frac{y}{2} = x\,2\cos^2\frac{y}{2} = 0.$$

By dividing both sides of this latter equation by $2\cos^2\frac{y}{2}$ we get

$$\frac{y'}{2\cos^2\frac{y}{2}}+\tan\frac{y}{2}+x=0.$$

The substitution $\tan\frac{y}{2}=z$, $\frac{dz}{dx}=\frac{y'}{2\cos^2\frac{y}{2}}$ reduces the last equation to the linear equation $\frac{dz}{dx}+z=-x$ whose general solution is

$$z = 1 - x + Ce^{-x}.$$

Replacing z by its expression in terms of y we obtain the general integral of the given equation

$$\tan\frac{y}{2}=1-x+Ce^{-x}. \quad \blacklozenge$$

In some equations the sought-for function $y(x)$ may be under the integral sign. In such cases it is sometimes possible to reduce a given equation to a differential one by differentiation.

Example 10. Solve the equation

$$x\int_0^x y(t)\,dt=(x+1)\int_0^x ty(t)\,dt \quad x>0.$$

Solution. Differentiating both sides of this equation with respect to x we get

$$\int_0^x y(t)\,dt + xy(x) = \int_0^x ty(t)\,dt + (x+1)\,xy(x)$$

or

$$\int_0^x y(t)\,dt = \int_0^x ty(t)\,dt + x^2y(x).$$

Differentiating once again with respect to x we obtain a homogeneous linear equation in $y(x)$:

$$y(x) = xy(x) + x^2y'(x) + 2xy(x)$$

or

$$x^2y'(x) + (3x - 1)\,y(x) = 0.$$

Separating the variables and integrating we find that

$$y = C\frac{1}{x^3}e^{-1/x}.$$

It is easy to verify that this solution satisfies the original equation.

Solve the following Bernoulli equations:

157. $y' + 2xy = 2xy^2$.

158. $3xy^2y' - 2y^3 = x^3$.

159. $(x^3 + e^y)\,y' = 3x^2$.

160. $y' + 2xy = y^2e^{x^2}$.

1 1. $y' - 2ye^x = 2\sqrt{ye^x}$.

162. $2y'\ln x + \frac{y}{x} = y^{-1}\cos x$.

163. $2y'\sin x + y\cos x = y^3\sin^2 x$.

164. $(x^2 + y^2 + 1)\,dy + xy\,dx = 0$.

165. $y' - y\cos x = y^2\cos x$.

By substituting the variables reduce the following non-linear equations to linear or Bernoulli equations and solve them:

166. $y' - \tan y = e^x \frac{1}{\cos y}$.

167. $y' = y\,(e^x + \ln y)$.

168. $y' \cos y + \sin y = x + 1$.

169. $yy' + 1 = (x - 1)\,e^{-\frac{y^2}{2}}$.

170. $y' + x \sin 2y = 2xe^{-x^2} \cos^2 y$.

Solve the following equations by differentiation:

171. $\int\limits_0^x ty(t)\,dt = x^2 y(x)$.

172. $y(x) = \int\limits_0^x y(t)\,dt + e^x$.

173. $\int\limits_a^x ty(t)\,dt = x^2 + y(x)$.

174. $\int\limits_0^1 y(xt)\,dt = ny(x)$.

7. Total differential equations. The integrating factor

7.1. Total differential equations. A differential equation f the form

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1}$$

is said to be *a total differential equation* if its left-hand side is a total differential of some function $u(x, y)$, that is,

$$M\,dx + N\,dy \equiv du \equiv \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

Theorem. *For equation* (1) *to be a total differential equation it is necessary and sufficient that the condition*

$$\frac{\partial M}{\partial y} \equiv \frac{\partial N}{\partial x} \tag{2}$$

should hold in some range D of the variables x and y.

The general integral of equation (1) is of the form $u(x, y) = C$ or

$$\int\limits_{x_0}^{x} M(x, y)\,dx + \int\limits_{y_0}^{y} N(x_0, y)\,dy = C. \tag{3}$$

Example 1. Solve the differential equation

$$(\sin xy + xy \cos xy)\, dx + x^2 \cos xy\, dy = 0.$$

Solution. We verify that this equation is a total differential equation:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(\sin xy + xy \cos xy) = x \cos xy + x \cos xy - x^2 y \sin xy = 2x \cos xy - x^2 y \sin xy,$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2 \cos xy) + 2x \cos xy - x^2 y \sin xy,$$

so that

$$\frac{\partial M}{\partial y} \equiv \frac{\partial N}{\partial x},$$

i.e. condition (2) is fulfilled. Thus the given equation is a total differential equation and

$$M = \frac{\partial u}{\partial x} = \sin xy + xy \cos xy, \quad N = \frac{\partial u}{\partial y} = x^2 \cos xy,$$

therefore

$$u(x, y) = \int (\sin xy + xy \cos xy)\, dx + \varphi(y),$$

where $\varphi(y)$ is as yet an undetermined function.

Integrating we get

$$u(x, y) = x \sin xy + \varphi(y).$$

The partial derivative $\frac{\partial u}{\partial y}$ of the obtained function $u(x, y)$ must equal $x^2 \cos xy$, which yields

$$x^2 \cos xy + \varphi'(y) = x^2 \cos xy,$$

whence $\varphi'(y) = 0$, so that $\varphi(y) = C$. Thus

$$u(x, y) = x \sin xy + C.$$

The general integral of the original differential equation is

$$x \sin xy = C. \ \blacklozenge$$

In integrating some differential equations it is possible for the terms to be grouped in such a way that easily integrable combinations should be obtained.

Example 2. Solve the differential equation

$$(x^3 + xy^2)\, dx + (x^2 y + y^3)\, dy = 0. \tag{4}$$

Solution. Here $\frac{\partial M}{\partial y} = 2xy$, $\frac{\partial N}{\partial x} = 2xy$, so that condition (2) is fulfilled and hence the given equation is a total differential equation. The equation is easy to reduce to the form $du = 0$ by immediate grouping of its terms. To the end we rewrite it like this:

$$x^3\,dx + xy\,(y\,dx + x\,dy) + y^3\,dy = 0.$$

Obviously,

$$x^3\,dx = d\left(\frac{x^4}{4}\right), \qquad xy\,(y\,dx + x\,dy) = xy\,d\,(xy) = d\left(\frac{(xy)^2}{2}\right),$$

$$y^3\,dy = d\left(\frac{y^4}{4}\right).$$

Therefore equation (4) may be written in the form

$$d\left(\frac{x^4}{4}\right) + d\left(\frac{(xy)^2}{2}\right) + d\left(\frac{y^4}{4}\right) = 0$$

or

$$d\left[\frac{x^4}{4} + \frac{(xy)^2}{2} + \frac{y^4}{4}\right] = 0.$$

Consequently,

$$x^4 + 2\,(xy)^2 + y^4 = C$$

is the general integral of equation (4).

Integrate the total differential equations below:

175. $x\,(2x^2 + y^2) + y\,(x^2 + 2y^2)\,y' = 0.$

176. $(3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0.$

177. $$\left(\frac{x}{\sqrt{x^2+y^2}} + \frac{1}{x} + \frac{1}{y}\right)dx + \left(\frac{y}{\sqrt{x^2+y^2}} + \frac{1}{y} - \frac{x}{y^2}\right)dy = 0.$$

178. $$\left(3x^2 \tan y - \frac{2y^3}{x^3}\right)dx + \left(x^3 \sec^2 y + 4y^3 + \frac{3y^2}{x^2}\right)dy = 0.$$

179. $$\left(2x + \frac{x^2+y^2}{x^2y}\right)dx = \frac{x^2+y^2}{xy^2}\,dy.$$

180. $$\left(\frac{\sin 2x}{y} + x\right)dx + \left(y - \frac{\sin^2 x}{y^2}\right)dy = 0.$$

181. $(3x^2-2x-y)\,dx+(2y-x+3y^2)\,dy=0.$

182. $\left(\frac{xy}{\sqrt{1+x^2}}+2xy-\frac{y}{x}\right)dx$

$$+(\sqrt{1+x^2}+x^2-\ln x)dy=0$$

183. $\frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}+\frac{x\,dy-y\,dx}{x^2}=0.$

184. $\left(\sin y+y\sin x+\frac{1}{x}\right)dx$

$$+\left(x\cos y-\cos x+\frac{1}{y}\right)dy=0.$$

185. $\frac{y+\sin x\cos^2 xy}{\cos^2 xy}\,dy+\left(\frac{x}{\cos^2 xy}+\sin y\right)dy=0.$

186. $\frac{2x\,dx}{y^3}+\frac{(y^2-3x^2)\,dy}{y^4}=0,\quad y|_{x=1}=1.$

187. $y\,(x^2+y^2+a^2)\,dy+x\,(x^2+y^2-a^2)\,dx=0.$

188. $(3x^2y+y^3)\,dx+(x^3+3xy^2)\,dy=0.$

7.2. The integrating factor. In some cases, where equation (1) is not a total differential equation, it may be possible to select a function $\mu(x, y)$ after multiplying by which the left-hand side of (1) turns into a total differential

$$du=\mu M\,dx+\mu N\,dy.$$

This function $\mu(x, y)$ is called *an integrating factor*. From the definition of the integrating factor we have

$$\frac{\partial}{\partial y}(\mu M)=\frac{\partial}{\partial x}(\mu N)$$

or

$$N\frac{\partial\mu}{\partial x}-M\frac{\partial\mu}{\partial y}=\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\mu,$$

whence

$$N\frac{\partial\ln\mu}{\partial x}-M\frac{\partial\ln\mu}{\partial y}=\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}. \qquad (5)$$

We have obtained a partial differential equation to find an integrating factor.

We shall point out some particular cases where it is relatively easy to find a solution of equation (5), i.e. to find an integrating factor.

1. If $\mu = \mu(x)$, then $\frac{\partial \mu}{\partial y} = 0$ and equation (5) will take the form

$$\frac{d \ln \mu}{dx} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}. \tag{6}$$

For an integrating factor independent of y to exist it is necessary and sufficient that the right-hand side of (6) should be a function of x alone. In such a case $\ln \mu$ can be found by a quadrature.

Example 3. Solve the equation $(x + y^2)\,dx - 2xy\,dy = 0$.

Solution. Here $M = x + y^2$, $N = -2xy$. We have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y + 2y}{-2xy} = -\frac{2}{x},$$

consequently,

$$\frac{d \ln \mu}{dx} = -\frac{2}{x}, \quad \ln \mu = -\ln|x|, \quad \mu = \frac{1}{x^2}.$$

The equation

$$\frac{x + y^2}{x^2}\,dx - 2\frac{xy}{x^2}\,dy = 0$$

is a total differential equation. Its left-hand side may be represented by

$$\frac{dx}{x} - \frac{2xy\,dy - y^2\,dx}{x^2} = 0, \quad \text{whence } d\left(\ln|x| - \frac{y^2}{x}\right) = 0$$

and the general integral of the given equation is

$x = C \cdot e^{y^2/x}$. ◆

2. Similarly, if $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\frac{1}{M}$ is a function of y alone, then equation (1) has an integrating factor $\mu = \mu(y)$ depending on y alone.

Example 4. Solve the equation $2xy \ln y\,dx + \left(x^2 + y^2\sqrt{y^2+1}\right)dy = 0$.

Solution. Here $M = 2xy \ln y$, $N = x^2 + y^2\sqrt{y^2+1}$. We have

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{2x - 2x(\ln y + 1)}{2xy \ln y} = -\frac{1}{y},$$

consequently,

$$\frac{d\ln\mu}{dy}=-\frac{1}{y}, \quad \mu=\frac{1}{y}.$$

The equation

$$\frac{2xy\ln y\,dx}{y}+\frac{x^2+y^2\sqrt{y^2+1}}{y}\,dy=0$$

is a total differential equation. It can be written as $d\left(x^2\ln y\right)+y\sqrt{y^2+1}\,dy=0$, whence

$$x^2\ln y+\frac{1}{3}\left(y^2+1\right)^{\frac{3}{2}}=C.$$

Example 5. Solve the equation $(3x+2y+y^2)\,dx+$ $+(x+4xy+5y^2)\,dy=0$ if its integrating factor is of the form $\mu=\varphi(x+y^2)$.

Solution. Put $z=x+y^2$, then $\mu=\varphi(z)$ and, consequently,

$$\frac{\partial\ln\mu}{\partial x}=\frac{d\ln\mu}{\partial z}\frac{\partial z}{\partial x}=\frac{d\ln\mu}{dz}, \quad \frac{\partial\ln\mu}{\partial y}=\frac{\partial\ln\mu}{dz}\frac{\partial z}{\partial y}=\frac{d\ln\mu}{dz}2y.$$

Equation (5) used to find the integrating factor will be of the form

$$(N-2My)\frac{d\ln\mu}{dz}=\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} \quad \text{or} \quad \frac{d\ln\mu}{dz}=\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N-2My}$$

Since $M=3x+2y+y^2$, $N=x+4xy+5y^2$, we have

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N-2My}=\frac{1}{x+y^2}=\frac{1}{z},$$

and hence $\frac{d\ln\mu}{dz}=\frac{1}{z}$, whence $\mu=z$, i.e. $\mu=x+y^2$. Multiplying the given equa ion by $\mu=x+y^2$ we get

$$(3x^2+2xy+4xy^2+2y^3+y^4)\,dx$$
$$+(x^2+4x^2y+6xy^2+4xy^3+5y^4)\,dy=0.$$

This is a total differential equation and according to (3) its general integral is

$$\int_{x_0}^{x} (3x^2+2xy+4xy^2+2y^3+y^4)\,dx+$$

$$+\int_{y_0}^{y} (x_0^2+4x_0^2y+6x_0y^2+4x_0y^3+5y^4)\,dy+C$$

or

$$x^3+x^2y+2x^2y^2+2xy^3+xy^4+y^5=\tilde{C},$$

where

$$\tilde{C}=C+x_0^2y_0+2x_0^2y_0^2+2x_0y_0^3+x_0y_0^4+y_0^5+x_0^3.$$

After some simple transformations we have

$$(x+y)(x+y^2)^2=\tilde{C}.$$

Integrate the following equations:

189. $(1-x^2y)\,dx+x^2(y-x)\,dy=0,\quad \mu=\varphi(x).$

190. $(x^2+y)\,dx-x\,dy=0,\quad \mu=\varphi(x).$

191. $(x+y^2)\,dx-2xy\,dy=0,\quad \mu=\varphi(x).$

192. $(2x^2y+2y+5)\,dx+(2x^3+2x)\,dy=0,\ \mu=\varphi(x).$

193. $(x^4\ln x-2xy^3)\,dx+3x^2y^2\,dy=0,\quad \mu=\varphi(x).$

194. $(x+\sin x+\sin y)\,dx+\cos y\,dy=0,\quad \mu=\varphi(x).$

195. $(2xy^2-3y^3)\,dx+(7-3xy^2)\,dy=0,\quad \mu=\varphi(y).$

196. $(3y^2-x)\,dx+(2y^3-6xy)\,dy=0,\quad \mu=\varphi(x+y^2).$

197. $(x^2+y^2+1)\,dx-2xy\,dy=0,\quad \mu=\varphi(y^2-x^2).$

198. $x\,dx+y\,dy+x(x\,dy-y\,dx)=0,\ \mu=\varphi(x^2+y^2).$

8. First-order differential equations not solved for the derivative

8.1. Equations of the first order and nth degree in y'. Suppose we are given the differential equation

$$(y')^n+p_1(x,y)(y')^{n-1}+\dots$$
$$\dots+p_{n-1}(x,y)\,y'+p_n(x,y)=0. \quad (1)$$

We solve this equation for y'. Let

$$y' = f_1(x, y), \quad y' = f_2(x, y), \ \dots$$
$$\dots, \quad y' = f_k(x, y) \quad (k \leqslant n)$$

be real solutions of equation (1).

The general integral of equation (1) will be expressed by a totality of integrals:

$$\Phi_1(x, y, C) = 0, \quad \Phi_2(x, y, C) = 0, \ \dots$$
$$\dots \ \Phi_k(x, y, C) = 0,$$

where $\Phi_i(x, y, C)$ is the integral of the equation $y' = f_i(x, y)$ $(i = 1, \dots, k)$.

Thus k integral curves pass through each point of the domain where y' takes on real values.

Example 1. Solve the equation $yy'^2 + (x - y)y' - x = 0$.

Solution. We solve this equation for y':

$$y' = \frac{y - x \pm \sqrt{(x-y)^2 + 4xy}}{2y}; \quad y' = 1, \quad y' = -\frac{x}{y}$$

whence

$$y = x + C, \quad y^2 + x^2 = C^2.$$

Example 2. Solve the equation $2y'^2 - 2xy' - 2y + x^2 = 0$.

Solution. We resolve the equation for y:

$$y = y'^2 - xy' + \frac{x_2}{2}.$$

Let us put $y' = p$, where p is a parameter; then we get

$$y = p^2 - xp + \frac{x^2}{2}. \tag{2}$$

Differentiating (2) we find that

$$dy = 2p\,dp - p\,dx - x\,dp + x\,dx.$$

But since $dy = p\,dx$, we have

$$p\,dx = 2p\,dp - p\,dx - x\,dp + x\,dx$$

or

$$2p\,dp - 2p\,dx - x\,dp + x\,dx = 0,$$

$$2p(dp - dx) - x(dp - dx) = 0, \quad (2p - x)(dp - dx) = 0.$$

Consider two cases: (1) $dp - dx = 0$, whence $p = x + C$, where C is an arbitrary constant. Substituting the value of p in (2) we obtain the general solution of the given equation:

$$y = Cx + C^2 + \frac{x^2}{2}. \tag{3}$$

In the equation $p = x + C$ one cannot replace p by y' and integrate the resulting equation $y' = x + C$ (there appearing another arbitrary constant, which is inadmissible since the differential equation considered is a first order equation).

(2) $2p - x = 0$, whence $p = x/2$. Substituting into (2) we obtain one more solution

$$y = x^2/4. \tag{4}$$

Let us see if the property of uniqueness is broken at each point of solution (4), i.e. if it is singular (see Sec. 11). To this end take on the integral curve (4) an arbitrary point $M_0(x_0, y_0)$, where $y_0 = x_0^2/4$. Now seek a solution, contained in the general solution (3), whose graph passes through the point M_0, $\left(x_0, \frac{x_0^2}{4}\right)$. Substituting the coordinates of this point into the general solution (3) we have

$$\frac{x_0^2}{4} = Cx_0 + C^2 + \frac{x_0^2}{2} \quad \text{or} \quad \left(C + \frac{x_0}{2}\right)^2 = 0,$$

whence $C = -x_0/2$. We substitute this value of the constant C into (3). Then we obtain the particular solution

$$y = \frac{x^2}{2} - \frac{x_0 x}{2} + \frac{x_0^2}{4} \tag{5}$$

which does not coincide with solution (4). We have $y' = x/2$, $y' = x - x_0/2$ for solutions (4) and (5) respectively. Both derivatives coincide when $x = x_0$. Consequently, the property of uniqueness does not hold at the point M_0, i.e. two integral curves with the same tangent pass through it. Since x_0 is arbitrary, uniqueness fails at each point of solution (4), and this means that it is singular.

Integrate the following equations:

199. $4y'^2 - 9x = 0$.

200. $y'^2 - 2yy' = y^2(e^{2x} - 1)$.

201. $y'^2 - 2xy' - 8x^2 = 0$.

202. $x^2y'^2 + 3xyy' + 2y^2 = 0$.

203. $y'^2 - (2x + y)\, y' + x^2 + xy = 0$.

204. $y'^3 + (x + 2)\, e^y = 0$.

205. $y'^3 = yy'^2 - x^2y' + x^2y = 0$.

206. $y'^2 - yy' + e^x = 0$.

207. $y'^2 - 4xy' + 2y + 2x^2 = 0$.

8.2. Equations of the form $f(y, y') = 0$ and $f(x, y') = 0$. If equations $f(y, y') = 0$ and $f(x, y') = 0$ are easily resolvable for y', then, resolving them, we obtain equations with variables separable.

Let us consider cases where these equations cannot be resolved for y'.

A. An equation of the form $f(y, y') = 0$ is resolvable for y:

$$y = \varphi(y').$$

We put $y' = p$, then $y = \varphi(p)$. Differentiating this equation and replacing dy by $p\, dx$ we get

$$p\, dx = \varphi'(p)\, dp,$$

whence

$$dx = \frac{\varphi'(p)}{p}\, dp \quad \text{and} \quad x = \int \frac{\varphi'(p)}{p}\, dp + C.$$

We obtain the general solution of the equation in the parametric form

$$x = \int \frac{\varphi'(p)}{p}\, dp + C, \quad y = \varphi(p). \tag{6}$$

Example 3. Solve the equation $y = a\left(\frac{dy}{dx}\right)^2 + b\left(\frac{dy}{dx}\right)^3$, a and b being constants.

Solution. We put $\frac{dy}{dx} = p$, then $y = ap^2 + bp^3$, $dy = 2ap\, dp + 3bp^2\, dp$ or $p\, dx = 2ap\, dp + 3bp^2\, dp$. Hence $dx = 2a\, dp + 3bp\, dp$ and $x = 2ap + \frac{3}{2}bp^2 + C$. The general solution is

$$x = 2ap + \frac{3}{2}bp^2 + C, \quad y = ap^3 + bp^3.$$

B. If an equation of the form $f(y, y') = 0$ is not resolvable (or is difficult to resolve) for both y and y' but allows

the expression of y and y' in terms of some parameter t:

$$y=\varphi(t), \quad p=\psi(t) \quad \left(p=\frac{dy}{dx}\right),$$ 1268 1

then we proceed as follows. We have $dy=p\,dx=\psi(t)\,dx$. On the other hand, $dy=\varphi'(t)\,dt$, so that $\psi(t)\,dx=\varphi'(t)\,dt$ and $dx=\frac{\varphi'(t)}{\psi(t)}dt$; hence

$$x=\int\frac{\varphi'(t)}{\psi(t)}\,dt.$$

Thus we obtain the general solution of the given differential equation in the parametric form

$$x=\int\frac{\varphi'(t)}{\psi(t)}\,dt+C, \quad y=\varphi(t).$$

Example 4. Solve the equation $y^{2/3}+(y')^{2/3}=1$.

Solution. We put $y=\cos^3 t$, $y'=p=\sin^3 t$,

$$dx=\frac{dy}{p}=\frac{-3\cos^2 t\sin t\,dt}{\sin^3 t}=-3\,\frac{\cos^2 t}{\sin^2 t}\,dt.$$

Hence

$$x=\int\left(3-\frac{3}{\sin^2 t}\right)dt=3t+3\cot t+C.$$

The general solution is

$$x=3t+3\cot t+C, \quad y=\cos^3 t.$$

C. An equation of the form $f(x, y')=0$. Let this equation be resolvable for x:

$$x=\varphi(y').$$

Putting $y'\equiv p$ we get $dx=\varphi'(p)\,dp$. But $dx=\frac{dy}{p}$ and, consequently, $\frac{dy}{p}=\varphi'(p)\,dp$, so that

$$dy=p\varphi'(p)\,dp \quad \text{and} \quad y=\int p\varphi'(p)\,dp+C.$$

Thus we find that

$$x=\varphi(p), \quad y=\int\varphi'(p)\,p\,dp+C$$

is the general solution of the equation in parametric form (p being a parameter).

Remark. In formulas (6) and (7) p cannot be considered a derivative, it is simply a parameter.

Example 5. Solve the equation $a\ \frac{dy}{dx}+b\left(\frac{dy}{dx}\right)^2=x.$

Solution. Put $\frac{dy}{dx} = p$, then

$x = ap + bp^2, \quad dx = a\,dp + 2bp\,dp,$

$dy = p\,dx = ap\,dp + 2bp^2\,dp, \ y = \frac{a}{2}\,p^2 + \ \ bp^3 + C.$

So

$x=ap+bp^2,\ y=\frac{a}{2}\,p^2+\frac{2}{3}\,bp^3+C$ is the general solution.

As in case B we may try to solve the equation $f\,(x,\ y') = 0$ by introducing a parameter t.

Integrate the following equations:

208. $y = y'^2\,e^{y'}$.

209. $y' = e^{y'/y}$.

210. $x = \ln y' + \sin y'$.

211. $x = y'^2 - 2y' + 2$.

212. $y = y'\ln y'$.

213. $y = (y' - 1)\,e^{y'}$.

214. $y'^2 x = e^{1/y'}$.

215. $x\,(1 + y'^2)^{3/2} = a$.

216. $y^{\frac{2}{5}} + y'^{\frac{2}{5}} = a^{\frac{2}{5}}$.

217. $x = y' + \sin y'$.

218. $y = y'\,(1 + y'\cos y')$.

219. $y = \arcsin y' + \ln\,(1 + y'^2)$.

8.3. The Lagrange and Clairaut equations. *The Lagrange equation* is of the form

$$y = x\varphi\,(y') + \psi\,(y').$$

Setting $y' = p$, differentiating with respect to x and replacing dy by $p\,dx$ we reduce this equation to a linear equation in x as a function of p. Finding the solution of this last equation $x = r\,(p,\ C)$ we obtain the general solution of the original equation in parametric form:

$$x = r\,(p,\ C) \quad y = r\,(p,\ C)\ \varphi\,(p) + \psi\,(p) \qquad (p \text{ being a parameter}).$$

In addition the Lagrange equation may have some singular solutions (see Sec. 11) of the form $y = \varphi\,(c)\,x + \psi\,(c)$, where c is the root of the equation $c = \varphi\,(c)$.

Example 6. Integrate the equation $y = 2xy' + \ln y'$.

Solution. We set $y' = p$, then $y = 2xp + \ln p$. Differentiating we find that

$$p\,\mathrm{d}x = 2p\,\mathrm{d}x + 2x\,\mathrm{d}p + \frac{\mathrm{d}p}{p},$$

whence $p\frac{\mathrm{d}x}{\mathrm{d}p} = -2x - \frac{1}{p}$ or $\frac{\mathrm{d}x}{\mathrm{d}p} = -\frac{2}{p}x - \frac{1}{p^2}$.

We have obtained a first order equation linear in x; solving it we find that

$$x = \frac{C}{p^2} - \frac{1}{p}.$$

Substituting the obtained value of x in the expression for y we finally get

$$x = \frac{C}{p^2} - \frac{1}{p}, \quad y = \ln p + \frac{2C}{p} - 2. \ \blacklozenge$$

The Clairaut equation is of the form

$$y = xy' + \psi(y').$$

The method of solution is the same as for the Lagrange equation. The general solution of the Clairaut equation is of the form

$$y = Cx + \psi(C).$$

The Clairaut equation may have also a singular solution which is obtained by eliminating p from the equations

$$y = xp + \psi(p), \ x + \psi'(p) = 0.$$

Example 7. Integrate the equation $y = xy' + \frac{a}{2y'}$ ($a = \text{const}$).

Solution. Setting $y' = p$ we get

$$y = xp + \frac{a}{2p}.$$

Differentiating this latter equation and replacing $\mathrm{d}y$ by $p\,\mathrm{d}x$ we find that

$$p\,\mathrm{d}x = p\,\mathrm{d}x + x\,\mathrm{d}p - \frac{a}{2p^2}\,\mathrm{d}p,$$

whence

$$\mathrm{d}p\left(x - \frac{a}{2p^2}\right) = 0.$$

Equating the first factor to zero we get $dp = 0$, whence $p = C$ and the general solution of the original equation is $y = Cx + \frac{a}{2C}$, a one-parameter family of straight lines. Equating the second factor to zero we have

$$x = a/2p^2.$$

Eliminating p from this equation and from the equation

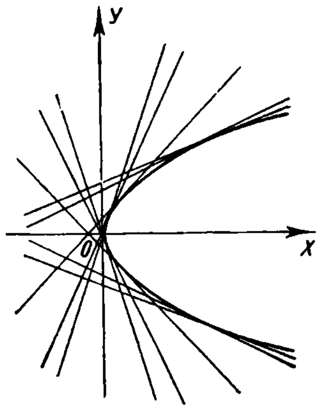

Fig. 14

$y = xp + \frac{a}{2p}$ we get $y^2 = 2ax$, which is also a solution of our equation (a singular one).

From a geometrical point of view the curve $y^2 = 2ax$ is the envelope of the family of straight lines given by the general solution (Fig. 14).

Integrate the following equations:

220. $y = 2xy' + \ln y'$.

221. $y = x(1 + y') + y'^2$.

222. $y = 2xy' + \sin y'$.

223. $y = xy'^2 - \frac{1}{y'}$.

224. $y = \frac{3}{2}xy' + e^{y'}$.

225. $y = xy' + \frac{a}{y'^2}$.

226. $y = xy' + y'^2$.

227. $xy'^2 - yy' - y' + 1 = 0$.

228. $y = xy' + a\sqrt{1 + y'^2}$.

229. $x = \frac{y}{y'} + \frac{1}{y'^2}$.

230. Find a curve each tangent of which forms with the coordinate axes a triangle of constant area $S = 2a^2$.

231. Find a curve for which the segment of a tangent to it contained between the coordinate axes is of constant length a.

9. The Riccati equation

A first order differential equation of the form

$$\frac{dy}{dx} + a(x)\,y^2 + b(x)\,y + c(x) = 0, \tag{1}$$

where $a(x)$, $b(x)$, and $c(x)$ are known functions, is called a (generalized) *Riccati equation.* If the coefficients a, b, and c are constant in a Riccati equation, then it allows separation of the variables and we at once obtain the general integral

$$C_1 - x = \int \frac{dy}{ay^2 + by + c}.$$

It was shown by Liouville that in the general case equation (1) cannot be integrated by quadratures.

The properties of the Riccati equation. 1°. *If we know some particular solution* $y_1(x)$ *of equation* (1), *then its general solution can be obtained by quadratures.*

Indeed, set

$$y = y_1(x) + z(x), \tag{2}$$

where $z(x)$ is a new unknown function. Substituting (2) into (1) we find that

$$\frac{dy_1}{dx} + \frac{dz}{dx} + a(x)\,(y_1^2 + 2y_1 z + z^2) + b(x)\,(y_1 + z) + c(x) = 0,$$

whence, by virtue of $y_1(x)$ being a solution of equation (1), we get

$$\frac{dz}{dx} + a(x)\,(2y_1 z + z^2) = b(x)\,z = 0$$

or

$$\frac{dz}{dx} + a(x)\,z^2 + [2a(x)\,y_1 + b(x)]\,z = 0. \tag{3}$$

Equation (3) is a particular case of the Bernoulli equation (see Sec. 6).

Example 1. Solve the Riccati equation

$$y' - y^2 + 2e^x y = e^{2x} + e^x \tag{4}$$

knowing its particular solution $y_1 = e^x$.

Solution. Setting $y = e^x + z(x)$ and substituting into equation (4) we get

$$\frac{dz}{dx} = z^2,$$

whence

$$-\frac{1}{z} = x - C \quad \text{or} \quad z = \frac{1}{C - x}.$$

Thus the general solution of equation (4) is

$$y = e^x + \frac{1}{C - x}.$$

Remark. The substitution

$$y = y_1(x) + \frac{1}{u(x)}$$

is often more advantageous in practice than substitution (2), for it immediately reduces the Riccati equation (1) to the linear form

$$u' - (2ay_1 + b)u = a.$$

2°. *If we know two particular solutions of equation* (1), *then its general integral can be found by one quadrature.*

Suppose that we know two particular solutions $y_1(x)$ and $y_2(x)$ of equation (1). Using the fact that we have the identity

$$\frac{dy_1}{dx} \equiv -a(x)y_1^2 - b(x)y_1 - c(x)$$

we write equation (1) in the form

$$\frac{1}{y - x_1} \frac{d(y - y_1)}{dx} = -a(x)(y + y_1) - b(x)$$

or

$$\frac{d}{dx}[\ln(y - y_1)] = -a(x)(y + y_1) - b(x). \tag{5}$$

Similarly for the second particular solution $y_2(x)$ we find that

$$\frac{d}{dx}[\ln(y - y_2)] = -a(x)(y + y_2) - b(x). \tag{6}$$

Subtracting equation (6) from equation (5) we get

$$\frac{d}{dx}\left[\ln\frac{y-y_1}{y-y_2}\right]=a(x)\,(y_2-y_1),$$

whence

$$\frac{y-y_1}{y-y_2}=C e^{\int a(x)[y_2(x)-y_1(x)]\,dx}. \tag{7}$$

Example 2. The equation

$$\frac{dy}{dx}=\frac{m^2}{x^4}-y^2, \quad m=\text{const}$$

has particular solutions

$$y_1=\frac{1}{x}=\frac{m}{x^2}, \quad y_2=\frac{1}{x}-\frac{m}{x^2}.$$

Using formula (7) we obtain the general integral of the original equation

$$\frac{y-y_1}{y-y_2}=C\,e^{-\int\frac{2m}{x^2}\,dx}, \quad \text{whence} \quad \frac{x^2y-x-m}{x^2y-x+m}=C\,e^{\frac{2m}{x}}.$$

Integrate the following Riccati equations using their particular solutions:

232. $y'e^{-x}+y^2-2ye^x=1-e^{2x}, \quad y_1=e^x.$

233. $y'+y^2-2y\sin x+\sin^2 x-\cos x=0,$
$y_1=\sin x.$

234. $xy'-y^2+(2x+1)\,y=x^2+2x, \quad y_1=x.$

235. $x^2y'=x^2y^2+xy+1, \quad y_1=-1/x.$

236. Find the general integral of the Riccati equation when the ratio of coefficients is independent of x, i.e. when $a(x):b(x):c(x)=m:n:p$ (m, n, and p are constants).

237. Prove that the Riccati equation preserves its form in any transformation of the independent variable $x=\varphi(t)$, where $\varphi(t)$ is any continuously differentiable function defined in the interval (t_0, t_1), with $\varphi'(t)\neq 0$ on (t_0, t_1).

10. Forming differential equation of families of lines. Problems in trajectories

10.1. Forming differential equations of families of lines. Suppose we are given an equation of a one-parameter family of plane curves

$$y = \varphi(x, a), \tag{1}$$

a being a parameter.

Differentiating (1) with respect to x we find that

$$y' = \varphi'_x(x, a). \tag{2}$$

Eliminating the parameter a from (1) and (2) we obtain the differential equation

$$F(x, y, y') = 0 \tag{3}$$

expressing a property common to all the curves of the family (1). Equation (3) is the desired differential equation of the family (1).

If a one-parameter family of curves is determined by the equation

$$\Phi(x, y, a) = 0$$

then the differential equation of that family is obtained by eliminating the parameter a from the equations

$$\begin{cases} \Phi(x, y, a) = 0 \\ \dfrac{\partial \Phi}{\partial x} + \dfrac{\partial \Phi}{\partial y} y' = 0. \end{cases}$$

Now suppose we have the relation

$$\Phi(x, y, a_1, a_2, \ldots, a_n) = 0, \tag{4}$$

where $a_1, a_2, \ldots, a_n$ are parameters. Differentiating (4) n times with respect to x and eliminating the parameters $a_1, a_2, \ldots, a_n$ from (4) and the equations obtained we arrive at a relation of the form

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0. \tag{5}$$

This is the differential equation of the given n-parameter family of lines (4) in the sense that (4) is the general integral of equation (5).

Example 1. Find the differential equation of the system of hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{1} = 1$.

Solution. Differentiating this equation with respect to x we get

$$\frac{2x}{a^2}-2yy'=0 \quad \text{or} \quad \frac{x}{a^2}=yy'.$$

We multiply both sides by x, then $x^2/a^2 = xyy'$. Substituting into the equation of the family we find that $xyy' - y^2 = 1$.

Example 2. Find the differential equation of the family of lines

$$y=a\left(1-e^{-\frac{x}{a}}\right), \ a \text{ being a parameter.}$$

Solution. We differentiate both sides of the equation with respect to x:

$$y' = e^{-x/a}.$$

From the expression for y' we find that $a=-\frac{x}{\ln y'}$ and, substituting this expression for a into the equation of the family of lines, we get

$$y=-\frac{x}{\ln y'}(1-y') \quad \text{or} \quad y\ln y'+x(1-y')=0.$$

Example 3. Set up the differential equation of the family of straight lines lying at a distance equal to unity away from the origin of coordinates.

Solution. We start from the normal equation of the straight line

$$x\cos\alpha + y\sin\alpha - 1 = 0, \tag{6}$$

where α is a parameter.

Differentiating (6) with respect to x we find that $\cos\alpha + y'\sin\alpha = 0$, whence $y' = -\cot\alpha$, consequently,

$$\sin\alpha=\frac{1}{\sqrt{1+y'^2}}, \quad \cos\alpha=-\frac{y'}{\sqrt{1+y'^2}}.$$

On substituting $\sin\alpha$ and $\cos\alpha$ into (6) we get

$$\frac{-xy'}{\sqrt{1+y'^2}}+\frac{y}{\sqrt{1+y'^2}}-1=0 \quad \text{or} \quad y=xy'+\sqrt{1+y'^2}.$$

Set up the differential equations of the following families of lines:

238. $y = a/x$.

239. $x^2 - y^2 = ax$.

240. $y = a\,e^{x/a}$.

241. $y = Cx - C - C^2$.

242. $y = e^x (ax + b)$.

243. $y^2 = 2Cx + C^2$.

244. $y = ax^2 + bx + c$.

245. $y = C_1 x + \frac{C_2}{x} + C_3$.

246. $(x - a)^2 + (y - b)^2 = 1$.

247. $y = C_1 e^x + C_2 e^{-x}$.

248. $y = a \sin (x + \alpha)$.

10.2. Problems in trajectories. Suppose we are given the family of plane curves

$$\Phi (x, y, a) = 0 \tag{7}$$

depending on a single parameter a.

A curve making at each of its points a fixed angle α with the curve of the family (7) passing through that point is called *an isogonal trajectory* of that family; if in particular $\alpha = \pi/2$, then it is called *an orthogonal trajectory*.

Assuming the family (7) to be given, we shall seek its isogonal trajectories.

A. Orthogonal trajectories. We set up the differential equation of the given family of curves (see Sec. 10.1). Let it be of the form

$$F (x, y, y') = 0.$$

The differential equation of the orthogonal trajectories is of the form

$$F \left(x, y -\frac{1}{y'}\right) = 0.$$

The general integral of this equation

$$\Phi_1 (x, y, C) = 0$$

gives the family of orthogonal trajectories.

Let the family of plane curves be given by the equation in polar coordinates

$$\Phi (\rho, \varphi, a) = 0, \tag{8}$$

where a is a parameter. Eliminating the parameter a from (8) and $\frac{\partial \Phi}{\partial \varphi} = 0$ we get the differential equation of the family (8): $F (\rho, \varphi, \rho') = 0$. Replacing ρ' by $-\rho^2/\rho'$ in it gives

the differential equation of the family of orthogonal curves

$$F\left(\rho, \varphi, -\frac{\rho^2}{\rho'}\right)=0.$$

B. Isogonal trajectories. Let trajectories intersect the curves of the given family at an angle α, with $\tan\alpha = k$. One can show that the differential equation of isogonal trajectories is of the form

$$F\left(x, y, \frac{y'-k}{1+ky'}\right)=0.$$

Example 4. Find the orthogonal trajectories of the family of lines $y = kx$.

Solution. The family of lines $y = kx$ consists of straight lines passing through the origin of coordinates. In order to find the differential equation of the given family we differentiate both sides of the equation $y = kx$ with respect to x. We have $y' = k$. Eliminating the parameter k from the system of equations

$$\begin{cases} y=kx, \\ y'=k \end{cases}$$

gives the differential equation of the family $xy' = y$. Replacing y' by $-1/y'$ we obtain the differential equation of the orthogonal trajectories $-x/y' = y$ or $yy' + x = 0$. The equation obtained is an equation with variables separable; integrating it we find the equation of the orthogonal trajectories $x^2 + y^2 = C$ $(C \geqslant 0)$. The orthogonal trajectories are circles with centre at the origin of coordinates (Fig. 15).

Example 5. Find the equation of the family of lines orthogonal to the family $x^2 + y^2 = 2ax$.

Solution. The given family of lines is a system of circles with centres on the Ox axis and tangent to the Oy axis.

Differentiating both sides of the equation of the given family with respect to x we find that $x + yy' = a$. Eliminating the parameter a from the equations $x^2 + y^2 = 2ax$ and $x + yy' = a$ gives the differential equation of the given family $x^2 - y^2 + 2xyy' = 0$. The differential equation of the orthogonal trajectories is

$$x_2-y^2+2xy\left(-\frac{1}{y'}\right)=0 \text{ or } y'=\frac{2xy}{x^2-y^2}.$$

This equation is homogeneous. Integrating it we find that $x^2 + y^2 = Cy$. The integral curves are circles with centres on the Oy axis and tangent to the Ox axis (Fig. 16).

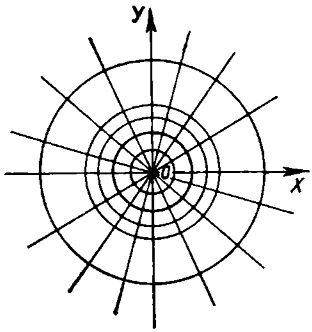

Fig. 15

Example 6. Find the orthogonal trajectories of the system of parabolas $y = ax^2$.

Solution. We set up the differential equation of the system of parabolas. To do this we differentiate both sides of the

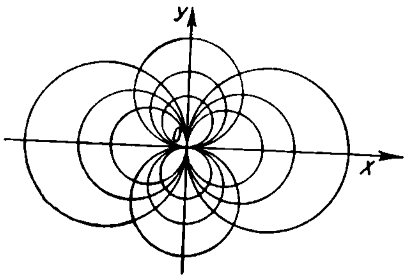

Fig. 16

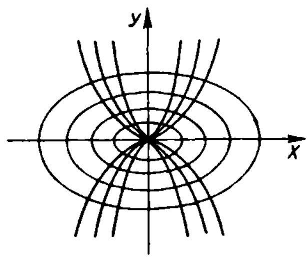

Fig. 17

given equation with respect to x: $y' = 2ax$. Eliminating the parameter a we find that $\frac{y'}{y} = \frac{2}{x}$ or $y' = \frac{2y}{x}$ is the differential equation of the given system. Replacing y' by $-1/y'$ in the equation gives the differential equation of the orthogonal trajectories

$$-\frac{1}{y'} = \frac{2y}{x} \text{ or } \frac{dy}{dx} = -\frac{x}{2y}.$$

Integrating we find that $y^2 = -\frac{x^2}{2} + C$ or $\frac{x^2}{2} + y^2 = C$, where $C > 0$. The orthogonal system is a system of ellipses (Fig. 17).

Example 7. Find the orthogonal trajectories of the system of lemniscates $\rho^2 = a \cos 2\varphi$.

Solution. We have

$$\rho^2 = a \cos 2\varphi, \quad \rho\rho' = -a \sin 2\varphi.$$

Eliminating the parameter a we obtain the differential equation of the given family of curves

$$\rho' = -\rho \tan 2\varphi.$$

Replacing ρ' by $-\rho^2/\rho'$ we find the differential equation of the family of orthogonal trajectories

$$-\rho^2/\rho' = -\rho \tan 2\varphi,$$

whence $\frac{d\rho}{\rho} = \cot 2\varphi \, d\varphi$. Integrating we find the equation of orthogonal trajectories

$$\rho^2 = C \sin 2\varphi.$$

The orthogonal trajectories of the system of lemniscates are

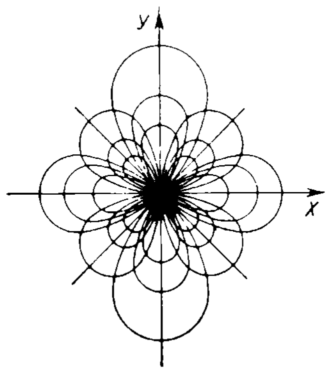

Fig. 18

lemniscates whose axes of symmetry make with the polar axis an angle of $\pm 45°$ (Fig. 18).

Find the orthogonal trajectories for the given families of curves:

249. $y^2 + 2ax = 0$, $a > 0$.

250. $y = ax^n$, a being a parameter.

251. $y = ae^{\sigma x}$, $\sigma = \text{const}$.

252. $\cos y = ae^{-x}$.

253. $x^2 + \frac{1}{2} y^2 = a^2$.

254. $x^2 - y^2 = a^2$.

255. $x^k + y^k = a^k$.

256. $x^2 + y^2 = 2ay$.

257. $x^2 - \frac{1}{3} y^2 = a^2$.

258. $\rho = a(1 + \cos\varphi)$.

259. $y^2 = 4(x - a)$.

11. Singular solutions of differential equations

The solution $y = \varphi(x)$ of the differential equation

$$F(x, y, y') = 0 \tag{1}$$

is said to be *singular* if the property of uniqueness is broken at each of its points, i.e. if besides this solution through each of its points (x_0, y_0) passes another solution which has at a point (x_0, y_0) the same tangent as the solution $y = \varphi(x)$ has but which does not coincide with it in an arbitrarily small neighbourhood of (x_0, y_0). We shall call the graph of a singular solution *the singular integral curve* of equation (1). If the function $F(x, y, y')$ and its partial derivatives $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial y'}$ are continuous in all the variables x, y, y', then any singular solution of equation (1) also satisfies the equation

$$\frac{\partial F(x, y, y')}{\partial y'} = 0. \tag{2}$$

Hence one must eliminate y' from equations (1) and (2) to find the singular solutions of equation (1).

The equation

$$\psi(x, y) = 0 \tag{3}$$

obtained on eliminating y' from (1) and (2) is called *the p-discriminant of equation* (1) and the curve determined by equation (3) is called a *p-discriminant curve* (PDC for short).

It often happens that a PDC splits into several branches. It is then necessary to determine whether each of the branches

is a distinct solution of equation (1) and, if it is, whether or not it is a singular solution, i.e. whether uniqueness is broken at each of its points.

Example 1. Find singular solutions of the differential equation

$$xy' + y'^2 - y = 0. \tag{4}$$

Solution. (a) We find the p-discriminant curve. In this case

$$F(x, y, y') \equiv xy' + y'^2 - y$$

and condition (2) takes the form

$$\frac{\partial F}{\partial y'} \equiv x + 2y' = 0,$$

hence $y' = -x/2$. Substituting this expression for y' into equation (4) we get

$$y = -\frac{x^2}{4}. \tag{5}$$

The curve (5) is the p-discriminant curve of equation (4): it consists of a single branch, a parabola.

(b) We ascertain whether the p-discriminant curve is a solution of the given equation. Substituting (5) and its derivative into (4) we see that $y = -x^2/4$ is a solution of equation (4).

(c) We ascertain whether the solution (5) is a singular solution of equation (4). To do this we find the general solution of equation (4). We rewrite (4) in the form $y = xy' + y'^2$. This is the Clairaut equation. Its general solution is

$$y = Cx + C^2. \tag{6}$$

We write out the conditions for the tangency of the two curves $y = y_1(x)$ and $y = y_2(x)$ at the point with the abscissa $x = x_0$:

$$y_1(x_0) = y_2(x_0), \qquad y_1'(x_0) = y_2'(x_0). \tag{7}$$

The first equation expresses the coincidence of the ordinates of the curves and the second the coincidence of the slopes of the tangents to those curves at the point with the abscissa $x = x_0$.

Setting $y_1(x) = -\frac{x^2}{4}$, $y_2(x) = Cx + C^2$, we find that conditions (7) take the form

$$-\frac{x_0^2}{4} = Cx_0 + C^2, \quad -\frac{x_0}{2} = C. \tag{8}$$

Substituting $C = -x_0/2$ into the first of the equations (8) we get

$$-\frac{x_0^2}{4} = -\frac{x_0^2}{2} + \frac{x_0^2}{4} \quad \text{or} \quad -\frac{x_0^2}{4} = -\frac{x_0^2}{4},$$

i.e. for $C = -x_0/2$ the first of the equations is satisfied identically, since x_0 is the abscissa of an arbitrary point.

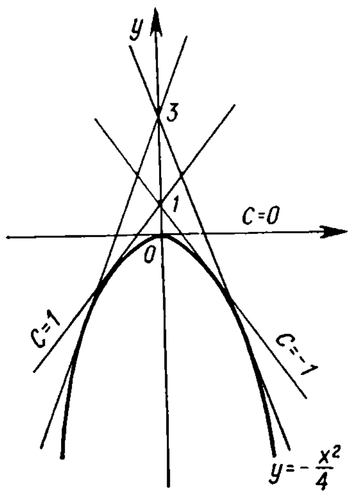

Fig. 19

So at each of its points the curve (5) is touched by some other curve of the family (6), namely by that curve for which $C = -x_0/2$. Hence $y = -x^2/4$ is a singular solution of equation (4).

(d) Geometrical interpretation.

The general solution of equation (4) is the family of straight lines (6) and the singular solution (5) is the envelope of that family of straight lines (Fig. 19). ◆

The envelope of the family of curves

$$\Phi(x, y, C) = 0 \tag{9}$$

is a curve which at each of its points is tangent to some curve of the family (9) and each segment of which is touched by an infinite number of curves of (9)*.

If (9) is the general integral of equation (1), then the envelope of the family of curves (9), if there is one, will be a singular integral curve of this equation. Indeed, at the points of the envelope the values of x, y, y' coincide with those of x, y, y' for the integral curve tangent to the envelope at a point (x, y) and, consequently, at each point of the envelope the values of x, y, y' satisfy the equation $F(x, y, y') = 0$, i.e. the envelope is an integral curve. Further, uniqueness is broken at each point of the envelope, since at least two integral curves pass through the points of the envelope in one direction: the envelope itself and the integral curve of the family (9) that is tangent to it at the point considered. Consequently, the envelope is a singular integral curve.

It is known from a course in mathematical analysis that the envelope forms a part of the C-discriminant curve (CDC for short) determined by the following system of equations

$$\begin{cases} \Phi(x, y, C) = 0, \\ \dfrac{\partial \Phi(x, y, C)}{\partial C} = 0, \end{cases} \tag{10}$$

Some branch of the CDC will a fortiori be the envelope if (1) it contains partial derivatives

$$\left|\frac{\partial \Phi}{\partial x}\right| \leqslant M, \quad \left|\frac{\partial \Phi}{\partial y}\right| \leqslant N \tag{11}$$

bounded in modulus, M and N being constants;

$$(2) \quad \frac{\partial \Phi}{\partial x} \neq 0 \ \text{ or } \ \frac{\partial \Phi}{\partial y} \neq 0. \tag{12}$$

Remark. Conditions (1) and (2) are only sufficient and therefore any branches of the CDC on which one of these conditions is violated may also be envelopes.

Example 2. Find singular solutions of the differential equation

$$xy'^2 - 2yy' + 4x = 0, \quad x > 0 \tag{13}$$

* We shall say that curves Γ_1 and Γ_2 touch at a point M_0 if they have a tangent in common at that point.

if we know its general integral

$$x^2 = C(y - C). \tag{14}$$

Solution. (a) We find the C-discriminant curve. We have

$$\Phi(x, y, C) \equiv C(y - C) - x^2$$

so that

$$\frac{\partial \Phi}{\partial C} \equiv y - 2C,$$

hence $C = y/2$. Substituting this value of C into (14) we get

$$x^2 \quad \frac{y}{2}\left(y - \frac{y}{2}\right),$$

whence

$$(y - 2x)(y + 2x) = 0 \quad \text{or} \quad y = \pm 2x. \tag{15}$$

This is the C-discriminant curve: it is made up of two straight lines $y = 2x$ and $y = -2x$.

(b) Making a direct substitution we see that each branch of the CDC is a solution of equation (13).

(c) Prove that each of the solutions (15) is a singular solution of equation (13). Indeed, since

$$\frac{\partial \Phi}{\partial x} = -2x, \quad \frac{\partial \Phi}{\partial y} = C$$

on each branch of the CDC we have

$$\left|\frac{\partial \Phi}{\partial x}\right| = |-2x| \leqslant 2b$$

(we assume that the solution $y(x)$ of equation (13) is considered on the interval $0 < a \leqslant x \leqslant b$)

$$\left|\frac{\partial \Phi}{\partial y}\right| = |C| \leqslant N; \quad \text{here } N = \max_{C \in G} |C|,$$

where G is the domain of allowed values of C.

Notice that on any branch of the CDC $\frac{\partial \Phi}{\partial x} = -2x \neq 0$ in the domain $x > 0$, so that one of the conditions (12) is fulfilled. Hence conditions (11) and (12) are fulfilled and, consequently, the straight lines (15) are the envelopes of the parabolas (14).

So we have established that each of the solutions (15) is a singular solution. ◆

In seeking singular solutions the following symbolic schemes are found to be useful:

$$\mathrm{PDC} \equiv \mathrm{E}\cdot\mathrm{S}\cdot\mathrm{T}^2 = 0, \tag{16}$$

$$\mathrm{CDC} \equiv \mathrm{E}\cdot\mathrm{N}^2\cdot\mathrm{S}^3 = 0. \tag{17}$$

Scheme (16) means that the equation of the p-discriminant curve may split into three equations:

1. E = 0, the equation of the envelope;
2. S = 0, the equation of the locus of spinodes (or cusps);
3. T = 0, the equation of the taclocus of integral curves, the factor T being contained in the PDC in the second power.

Scheme (17) means that the equation of the C-discriminant curve may split into the three equations:

1. E = 0, the equation of the envelope;
2. N = 0, the equation of the locus of nodes, the factor N being contained in the CDC in the second power.
3. S = 0, the equation of the locus of spinodes, the factor S being contained in the CDC in the third power.

It is not obligatory that all the component parts of the PDC and CDC should appear in relations (16) and (17).

Of all the loci only the envelope is a singular solution of a differential equation. The seeking of the envelope is simplified by the fact that it is contained in the first power in schemes (16) and (17).

As to the other loci (spinodes, nodes and tacpoints) an additional analysis is required in each particular case. The fact that a certain factor is contained in the PDC in the second power (and is not contained at all in the CDC) indicates that there may be a taclocus of integral curves here. Similarly, if a certain factor is contained in the CDC in the second power (and is not at all contained in the PDC), then there may be a locus of nodes here. Finally, if a certain factor is contained in the PDC in the first power and figures in the third power in the CDC, then the locus of spinodes may be present.

Example 3. Find singular solutions of the differential equation

$$2y\,(y' + 2) - xy'^2 = 0. \tag{18}$$

Solution. The singular solution, if there is one, is determined by the system

$$\begin{cases} 2y(y'+2)-xy'^2=0, \\ 2y-2xy'=0, \end{cases} \tag{19}$$

where the second equation of (19) is obtained from (18) by differentiation with respect to y'. On eliminating y' we obtain the p-discriminant curve

$$y^2+4xy=0$$

which splits into two branches

$$y=0 \tag{20}$$

$$y=-4x. \tag{21}$$

Substitution convinces us that both functions are solutions of equation (18).

In order to ascertain whether or not solutions (20) and (21) are singular we find the envelope of the family

$$Cy-(C-x)^2=0 \tag{22}$$

which is the general integral for (18).

We write out the system for determining the C-discriminant curve

$$\begin{cases} Cy-(C-x)^2=0, \\ y-2(C-x)=0, \end{cases}$$

whence by eliminating C we get

$$y^2+4xy=0 \text{ or } y=0 \text{ and } y=-4x$$

which coincides with (20) and (21). By virtue of the fact that conditions (11) and (12) hold on the lines (20) and (21) we conclude that the lines $y=0$ and $y=-4x$ are the envelopes and hence (20) and (21) are singular solutions of the given equation.

The integral curves (22) are the parabolas $y=\frac{(C-x)^2}{C}$ and the lines $y=0$ and $y=-4x$ are the envelopes of this system of parabolas (Fig. 20).

Example 4. Find singular solutions of the following differential equation

$$y'^2=4x^2. \tag{23}$$

Solution. We differentiate (23) with respect to y':

$$2y' = 0. \tag{24}$$

Eliminating y' from (23) and (24) we get $x^2 = 0$. The discriminant curve is the axis of ordinates. It is not the integral

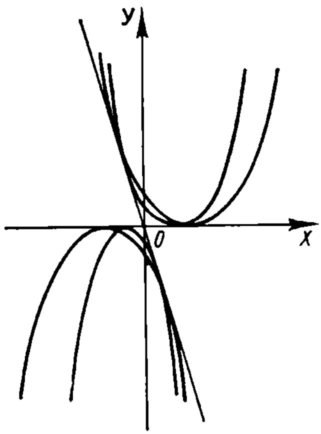

Fig. 20

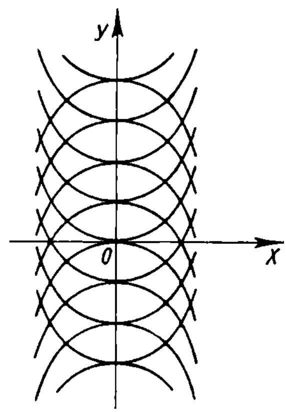

Fig. 21

curve of equation (23), but according to scheme (16) it may be the taclocus of integral curves.

The solutions of equation (23) are the parabolas

$$y = x^2 + C, \qquad y = -x^2 + C$$

and those smooth curves which can be composed of parts of them (Fig. 21).

It is seen from the drawing that the straight line $x = 0$ is the taclocus of the integral curves of equation (23).

Example 5. Find singular solutions of the differential equation

$$y'^2 (2 - 3y)^2 = 4 (1 - y). \tag{25}$$

Solution. Let us find the PDC. Eliminating y' from the system of equations

$$\begin{cases} y'^2 (2-3y)^2 - 4 (1-y) = 0, \\ 2y' (2-3y)^2 = 0 \end{cases}$$

we get

$$(2-3y)^2(1-y)=0. \tag{26}$$

Converting equation (25) to the form

$$\frac{dx}{dy}=\pm\frac{2-3y}{2\sqrt{1-y}}$$

we find its general integral

$$y^2(1-y)=(x-C)^2.$$

Let us find the CDC. Eliminating C from the system of equations

$$\begin{cases} y^2(1-y)-(x-C)^2=0, \\ 2(x-C)=0 \end{cases}$$

we have

$$y^2(1-y)=0. \tag{27}$$

So from (26) and (27) we have

$$\text{PDC} \equiv (1-y)(2-3y)^2=0,$$

$$\text{CDC} \equiv (1-y)\,y^2=0.$$

The factor $1-y$ is contained in the p-discriminant curve and in the C-discriminant in the first power and gives the

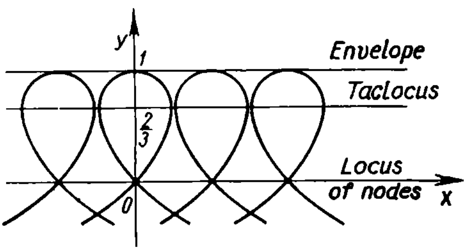

Fig. 22

envelope, i.e. the function $y=1$ is a singular solution of the differential equation (25). A direct substitution convinces us that $y=1$ does in fact satisfy the equation.

The equation $2-3y=0$, contained in the second power in the p-discriminant and not contained at all in the C-discriminant, gives the taclocus (T^2).

Finally, the equation $y = 0$, contained in the C-discriminant in the second power and not contained at all in the p-discriminant, gives the locus of nodes (N^2) (Fig. 22).

Example 6. Find singular solutions of the differential equation

$$3y = 2xy' - \frac{2}{x} y'^2. \tag{28}$$

Solution. (a) We seek the p-discriminant curve. Differentiating (28) with respect to y' we get

$$0 = 2x - \frac{4}{x} y', \text{ whence } y' = \frac{x^2}{2}. \tag{29}$$

Substituting (29) into (28) we find the equation of the PDC:

$$\text{PDC} \equiv 6y - x^3 = 0. \tag{30}$$

(b) We seek the general integral of equation (28). Denoting y' by p we rewrite (28) as

$$3y = 2xp - \frac{2}{x} p^2. \tag{31}$$

Differentiating both sides of (28) with respect to x and taking into account the fact that $y' = p$ we have

$$px^2 - 2p^2 = (2x^3 - 4px)\frac{\mathrm{d}p}{\mathrm{d}x}, \text{ whence}$$

$$(x^2 - 2p)\left(p - 2x\frac{\mathrm{d}p}{\mathrm{d}x}\right) = 0.$$

Equating the first factor to zero $x^2 - 2p = 0$ we get (29) and the relation $p - 2x\frac{\mathrm{d}p}{\mathrm{d}x} = 0$ gives

$$Cx = p^2. \tag{32}$$

Eliminating the parameter p from equations (31) and (32) we find the general solution of equation (28):

$$(3y + 2C)^2 = 4Cx^3. \tag{33}$$

(c) We find the C-discriminant curve. Differentiating (33) with respect to C we have

$$2C = x^3 - 3y. \tag{34}$$

Substituting (34) into (33) we obtain the equation of the CDC:

$$\text{CDC} \equiv (6y - x^3)\, x^3 = 0.$$

By the symbolic schemes (16) and (17) we conclude that $6y - x^3 = 0$ is the envelope of the system of semicubical parabolas (33) and $x = 0$ is the locus of spinodes (the factor x is contained in the equation of the CDC in the third power) (Fig. 23). Substitution into equation (28) convinces us

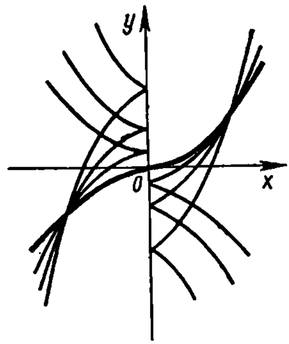

Fig. 23

that $y = x^3/6$ is a solution and that $x = 0$ is not a solution (equation (28) is meaningless when $x = 0$). Thus the solution $y = x^3/6$ is singular (the envelope of the family of integral curves).

In the following examples find singular solutions if any:

260. $(1 + y'^2)\, y^2 - 4yy' - 4x = 0$.

261. $y'^2 - 4y = 0$.

262. $y'^3 - 4xyy' + 8y^2 = 0$.

263. $y'^2 - y^2 = 0$.

264. $y' = \sqrt[3]{y^2} + a$. For what values of the parameter a does this equation have a singular solution?

265. $(xy' + y)^2 + 3x^5 (xy' - 2y) = 0$.

266. $y\,(y - 2xy')^2 = 2y'$.

267. $8y'^3 - 12y'^2 = 27\,(y - x)$.

268. $(y' - 1)^2 = y^2$.

Using the C-discriminant find singular solutions of the first order differential equations below when we know their general integrals.

269. $y=y'^2-xy'+\frac{x^2}{2}, \quad y=Cx+C^2+\frac{x^2}{2}.$

270. $(xy'+y)^2=y^2y', \quad y(C-x)=C^2.$

271. $y^2y'^2+y^2=1, \quad (x-C)^2+y^2=1.$

272. $y'^2-yy'+e^x=0, \quad y=Ce^x+\frac{1}{C}.$

273. $3xy'^2-6yy'+x+2y=0, \quad x^2+C(x-3y)+C^2=0.$

274. $y=xy'+\sqrt{a^2y'^2+b^2}, \quad y=Cx+\sqrt{a^2C^2+b^2}.$

12. Miscellaneous problems

Integrate the following equations:

275. $y'=(x-y)^2+1.$

276. $x\sin x\cdot y'+(\sin x-x\cos x)\,y=\sin x\cos x-x.$

277. $\frac{dy}{dx}+y\cos x=y^n\sin 2x, \ n\neq 1.$

278. $(x^3-3xy^2)\,dx+(y^3-3x^2y)\,dy=0.$

279. $(5xy-4y^2-6x^2)\,dx+(y^2-8xy+2.5x^2)\,dy=0$

280. $(3xy^2-x^2)\,dx+(3x^2y-6y^2-1)\,dy=0.$

281. $(y-xy^2\ln x)\,dx+x\,dy=0, \quad \mu=\varphi(x\cdot y).$

282. $(2xy\,e^{x^2}-x\sin x)\,dx+e^{x^2}\,dy=0.$

283. $y'=\frac{1}{2x-y^2}.$

284. $x^2+xy'=3x+y'.$

285. $xyy'-y^2=x^4.$

286. $\frac{dx}{x^2-xy+y^2}=\frac{dy}{2y^2-xy}.$

287. $(2x-1)\,y'-2y=\frac{1-4x}{x^2}.$

288. $(x-y+3)\,dx+(3x+y+1)\,dy=0.$

289. $y' + \cos\frac{x+y}{2} = \cos\frac{x-y}{2}$.

290. $y'(3x^2 - 2x) - y(6x - 2) = 0$.

291. $xy^2y' - y^3 = \frac{1}{3}x^4$.

292. $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0, \quad y\Big|_{x=1} = 1$.

293. $(x^2 + y^2)\,dx - xy\,dy = 0$.

294. $(x - y + 2)\,dx + (x - y + 3)\,dy = 0$.

295. $(xy^2 + y)\,dx - x\,dy = 0$.

296. $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$.

297. $(x - 1)(y^2 - y + 1)\,dx = (y + 1)(x^2 + x + 1)\,dy$.

298. $(x - 2xy - y^2)\,y' + y^2 = 0$.

299. $y\cos x\,dx + (2y - \sin x)\,dy = 0$.

300. $y' - 1 = e^{x+2y}$.

301. $2(x^5 + 2x^3y - y^2x)\,dx + (y^2 + 2x^2y - x^4)\,dy = 0$.

302. $x^2y^ny' = 2xy' - y, \quad n \neq -2$.

303. $[3(x + y) + a^2]\,y' = 4(x + y) + b^2$.

304. $(x - y^2)\,dx + 2xy\,dy = 0$.

305. $xy' + y = y^2 \ln x, \quad y|_{x=1} = \frac{1}{2}$.

306. $\sin(\ln x)\,dx - \cos(\ln y)\,dy = 0$.

307. $y' = \sqrt{\frac{9y^2 - 6y + 2}{x^2 - 2x + 5}}$.

308. $(5x - 7y + 1)\,dy + (x + y - 1)\,dx = 0$.

309. $(x + y + 1)\,dx + (2x + 2y - 1)\,dy = 0$,
$y|_{x=1} = 2$.

310. $y^3\,dx + 2(x^2 - xy^2)\,dy = 0$.

311. $y' = 2\left(\frac{y+2}{x+y-1}\right)^2$.

312. Show that a curve symmetrical with respect to the centre O (0, 0) to the integral curve of the equation $4x^2y'^2 - y^2 = xy^3$ is also an integral curve of this equation.

313. Find the integral curves of the differential equation $y' + xy'^2 - y = 0$ that are straight lines.

314. Find a curve if it is known that the area of the region bounded by the coordinate axes, this curve and the ordinate of any point on it is equal to the cube of that ordinate.

315. The area bounded by a curve, the axes of coordinates and the ordinate of some point of the curve is equal to the length of the corresponding arc of the curve. Find the equation of this curve if it is known that it passes through the point $M(0, 1)$.

CHAPTER II

DIFFERENTIAL EQUATIONS OF HIGHER ORDERS

13. Basic concepts and definitions

A differential equation of order n is of the form

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

or, if solved for $y^{(n)}$, of the form

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}). \tag{1}$$

The problem of finding a solution $y = \varphi(x)$ of equation (1) satisfying the initial conditions

$$y|_{x=x_0} = y_0, \quad y'|_{x=x_0} = y'_0, \quad \ldots, \quad y^{(n-1)}|_{x=x_0} = y_0^{(n-1)} \tag{2}$$

is called *the Cauchy*, or *initial value, problem* for equation (1).

The existence and uniqueness theorem for the initial value problem. *If in equation* (1) *the function* $f(x, y, y', y'', \ldots y^{(n-1)})$

(a) *is continuous with respect to all independent variables* $x, y, y', y'', \ldots, y^{(n-1)}$ *in some domain* D, and

(b) *has partial derivatives* $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial y'}, \frac{\partial f}{\partial y''}, \ldots, \frac{\partial f}{\partial y^{(n-1)}}$ *bounded in the domain* D *in the variables* $y, y', y'', \ldots, y^{(n-1)}$, *then there is an interval* $x_0 - h < x < x_0 + h$ *on which there exists a unique solution* $y = \varphi(x)$ *of equation* (1) *satisfying the conditions*

$$y|_{x=x_0} = y_0, \quad y'|_{x=x_0} = y'_0, \quad \ldots, \quad y^{(n-1)}|_{x=x_0} = y_0^{(n-1)},$$

where the values $x = x_0$, $y = y_0$, $y' = y'_0, \ldots, y^{(n-1)} = y_0^{(n-1)}$ *are contained in the domain* D.

For the second order equation $y'' = f(x, y, y')$ the initial conditions are of the form

$$y|_{x=x_0} = y_0, \quad y'|_{x=x_0} = y'_0,$$

where x_0, y_0, y'_0 are given numbers. In this case the existence and uniqueness theorem *geometrically* means that a unique

curve passes through a given point $M_0(x_0, y_0)$ of the xOy plane with a given slope of a tangent y_0'.

Consider, for example, the equation $y'' = \sin y' + e^{-x^2 y}$ and the initial conditions

$$y|_{x=x_0} = y_0, \quad y'|_{x=x_0} = y_0'.$$

In this case $f(x, y, y') \equiv \sin y' + e^{-x^2 y}$. This function is defined and continuous for all values of x, y, y'. Its partial derivatives with respect to y and y' are respectively equal to

$$\frac{\partial f}{\partial y} = -x^2 e^{-x^2 y}, \quad \frac{\partial f}{\partial y'} = \cos y'$$

and are everywhere continuous and bounded functions of their variables. Consequently, whatever the initial conditions

$$y|_{x=x_0} = y_0, \quad y'|_{x=x_0} = y_0'$$

there is a unique solution of the given equation satisfying these conditions. ◆

The general solution of an nth order differential equation (1) is the set of all of its solutions defined by the formula $y = \varphi(x, C_1, C_2, \ldots, C_n)$ containing n arbitrary constants $C_1, C_2, \ldots, C_n$ such that, if the initial conditions (2) are given, values $\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_n$ can be found such that $y = \varphi(x, \tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_n)$ is a solution of equation (1) satisfying these initial conditions.

Any solution obtained from the general solution for particular values of arbitrary constants $C_1, C_2, \ldots, C_n$ is called *a particular solution* of the differential equation (1).

An equation of the form $\Phi(x, y, C_1, C_2, \ldots, C_n) = 0$ which implicitly determines the general solution of a differential equation is called *the general integral of the equation.* Giving the constants $C_1, C_2, \ldots, C_n$ particular allowed numerical values we obtain *a particular integral* of the differential equation. The graph of a particular solution or of a particular integral is called *an integral curve* of the given differential equation.

Example 1. Show that $y = C_1 x + C_2$ is the general solution of the differential equation $y'' = 0$.

Solution. We show that $y = C_1 x + C_2$ satisfies the given equation for any values of the constants C_1 and C_2. Indeed we have $y' = C_1$, $y'' = 0$.

Now let $y\,|_{x=x_0} = y_0$, $y'\,|_{x=x_0} = y'_0$ be arbitrary initial conditions. We show that the constants C_1 and C_2 can be selected so that $y = C_1x + C_2$ satisfies these conditions. We have $y = C_1x + C_2$, $y' = C_1$. Setting $x = x_0$ we obtain the system

$$\begin{cases} y_0 = C_1x_0 + C_2, \\ y'_0 = C_1 \end{cases}$$

from which it can be uniquely determined that $C_1 = y'_0$ and that $C_2 = y_0 - x_0y'_0$. Thus the solution $y = y'_0(x-x_0) + y_0$ satisfies the initial conditions set. Geometrically this means that a unique curve passes through each point $M_0(x_1, y_0)$ of the xOy plane with a given slope y'_0.

Giving a single initial condition, for example $y\,|_{x=x_0} = y_0$, determines a bundle of straight lines with centre at a point $M_0(x_0, y_0)$, i.e. one initial condition is insufficient to separate out a unique solution.

316. The differential equation $y'' = 2\sqrt{y'}$ has two solutions $y_1(x) \equiv 0$, $y_2(x) \equiv x^3/3$ satisfying the initial conditions $y\,|_{x=0} = 0$, $y'\,|_{x=0} = 0$. Why does the result not contradict the existence and uniqueness theorem for the initial value problem?

317. Can the graphs of two solutions of a given equation touch at some point (x_0, y_0) in the xOy plane (a) for the equation $y' = x^2 + y^2$; (b) for the equation $y'' = x^2 + y^2$; (c) for the equation $y''' = x^2 + y^2$?

In the problems below show that the given functions are solutions of the indicated equations:

318. $y = x(\sin x - \cos x)$, $\quad y'' + y = 2(\cos x + \sin x)$.

319. $y = x^2 \ln x$, $\quad xy''' = 2$.

320. $x + C = e^{-y}$, $\quad y'' = y'^2$.

321. $\begin{cases} x = 1 + e^t, \\ y = te^t, \end{cases} \quad (x-1)\,y'' = 1.$

322. $\begin{cases} x = C_1 + \dfrac{t^4}{4}, \\ y = C_2 + \dfrac{t^5}{5}, \end{cases} \quad y'y'^3 = 1.$

Show that the given functions are the general solutions of the corresponding equations.

323. $y = C_1 \sin x + C_2 \cos x, \quad y'' + y = 0.$

324. $y = C_1 e^x + C_2 e^{2x} + 1, \quad y'' - 3y' + 2y = 2.$

Show that the relations given below are the integrals (general or particular) of the indicated equations.

325. $(x - C_1)^2 + (y - C_2)^2 = 1$

$y'' = (1 + y'^2)^{3/2}.$

326. $y^2 = 1 + (1 - x)^2, \quad y'^2 + yy'' = 1.$

14. Differential equations admitting of depression of their order

We shall show some forms of differential equations admitting of depression of order.

I. An equation of the form $y^{(n)} = f(x)$. After n-fold integration we obtain the general solution

$$y = \underbrace{\int \dots \int}_{n} f(x) \underbrace{dx \dots dx}_{n} + C_1 \frac{x^{n-1}}{(n-1)!} + C_2 \frac{x^{n-2}}{(n-2)!} + \dots + C_{n-1}x + C_n.$$

II. The equation does not contain the desired function or its derivatives to order $k - 1$ inclusively:

$$F(x, y^{(k)}, y^{(k+1)}, \dots, y^{(n)}) = 0. \tag{1}$$

The order of such an equation can be depressed by k units using the substitution $y^{(k)}(x) = p(x)$. Then equation (1) takes the form

$$F(x, p, p', \dots, p^{(n-k)}) = 0.$$

From this last equation we determine, if possible, $p = f(x, C_1, C_2, \dots, C_{n-k})$ and then find y from the equation $y^{(k)} = f(x, C_1, C_2, \dots, C_{n-k})$ by k-fold integration.

III. The equation contains no independent variable:

$$F(y, y', y'', \dots, y^{(n)}) = 0.$$

The substitution $y' = p$ makes it possible to depress the order of the equation by unity. In this case p is regarded as a new unknown function of y: $p = p(y)$. All the deriva-

tives $y', y'', \ldots, y^{(n)}$ are expressed in terms of the derivatives of the new unknown function p with respect to y:

$$y' = \frac{dx}{dx} = p,$$

$$y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy},$$

$$y''' = \frac{d}{dx}\left(p\frac{dp}{dy}\right) = \frac{d}{dy}\left(p\frac{dp}{dy}\right)\frac{dy}{dx} = p^2\frac{d^2p}{dy^2} + p\left(\frac{dp}{dy}\right)^2, \text{ etc.}$$

Substituting these expressions for $y', y'', \ldots, y^{(n)}$ in the equation we obtain an $(n-1)$th order differential equation.

IV. The equation $F(x, y, y', \ldots, y^{(n)}) = 0$ homogeneous in the variables $y, y', y'', \ldots, y^{(n)}$, i.e.

$$F(x, ty, ty', \ldots, ty^{(n)}) = t^k F(x, y, y', \ldots, y^{(n)}).$$

The order of such an equation can be depressed by unity using the substitution $y = e^{\int z\,dx}$, where z is a new unknown function of x: $z = z(x)$.

V. The equation written in differentials

$$F(x, y, dx, dy, d^2y, \ldots, d^{(n)}y) = 0$$

in which the function F is homogeneous in all its variables $x, y, dx, dy, d^2y, \ldots, d^ny$ if x and dx are assumed to be of the first degree and y, dy, d^2y, etc., of degree m. Then $\frac{dy}{dx}$ is of degree $m-1$, $\frac{d^2y}{dx^2}$ is of degree $m-2$, etc.

To depress the order the substitution $x = e^t$, $y = ue^{mt}$ is used. As a result we obtain a differential equation between u and t explicitly containing no t, i.e. admitting of depressing the order by unity (Case III).

Consider examples for the various cases of depressing the order of differential equations.

Example 1. Find the general solution of the equation $y''' = \sin x + \cos x$.

Solution. Integrating successively the equation we have

$$y'' = -\cos x + \sin x + C_1,$$

$$y' = -\sin x - \cos x + C_1x + C_2,$$

$$y = \cos x - \sin x + C_1\frac{x^2}{2} + C_2x + C_3.$$

Example 2. Find the general solution of the equation $y''' = \frac{\ln x}{x^2}$ and separate out the solution satisfying the initial conditions $y|_{x=1} = 0$, $y'|_{x=1} = 1$, $y''|_{x=1} = 2$.

Solution. We integrate this equation three times in succession:

$$y'' = \int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C_1,$$

$$y' = -\frac{1}{2}\ln^2 x - \ln x + C_1 x + C_2, \tag{2}$$

$$y = -\frac{x}{2}\ln^2 x + C_1\frac{x^2}{2} + C_2 x + C_3.$$

We find the solution satisfying the given initial conditions. Substituting the initial data $y|_{x=1} = 0$, $y'|_{x=1} = 1$, $y''|_{x=1} = 2$ in (2) we have

$$\frac{C_1}{2} + C_2 + C_3 = 0, \quad C_1 + C_2 = 1, \quad -1 + C_1 = 2.$$

Hence $C_1 = 3$, $C_2 = -2$, $C_3 = 1/2$. The desired solution is

$$y = -\frac{x}{2}\ln^2 x + \frac{3}{2}x^2 - 2x + \frac{1}{2}.$$

Example 3. Solve the equation $y''' = \sqrt{1 + (y'')^2}$.

Solution. The equation does not contain the desired function y or its derivative, therefore we set $y'' = p$. This done, the equation assumes the form

$$\frac{dp}{dx} = \sqrt{1 + p^2}.$$

Separating the variables and integrating, we find that

$$p = \frac{e^{x+C_1} - e^{-(x+C_1)}}{2}.$$

We replace p by y'':

$$y'' = \frac{e^{x+C_1} - e^{-(x+C_1)}}{2}.$$

Integrating successively we have

$$y' = \frac{e^{x+C_1} + e^{-(x+C_1)}}{2} + C_2 \text{ and } y = \frac{e^{x+C_1} - e^{-(x+C_1)}}{2} + C_2 x + C_3$$

or

$$y = \sinh(x + C_1) + C_2 x + C_3.$$

Example 4. Solve the equation $xy^{V} - y^{IV} = 0$.

Solution. The equation does not contain the desired function or its derivatives to the third order inclusively. Therefore setting $y^{IV} = p$ we get

$$x\frac{dp}{dx} - p = 0 \text{ whence } p = C_1 x, \ y^{IV} = C_1 x.$$

Integrating successively we find that

$$y''' = C_1\frac{x^2}{2} + C_2, \quad y'' = C_1\frac{x^3}{6} + C_2 x + C_3,$$

$$y' = C_1\frac{x^4}{24} + C_2\frac{x^2}{2} + C_3 x + C_4,$$

$$y = C_1\frac{x^5}{120} + C_2\frac{x^3}{6} + C_3\frac{x^2}{2} + C_4 x + C_5$$

or

$$y = \overline{C}_1 x^5 + \overline{C}_2 x^3 + \overline{C}_3 x^2 + C_4 x + C_5,$$

where $\overline{C}_1 = C_1/120$, $\overline{C}_2 = \overline{C}_2/6$, $\overline{C}_3 = \overline{C}_3/2$.

Example 5. Solve the equation $y'' + y'^2 = 2e^{-y}$.

Solution. The equation does not contain the independent variable x. Setting $y' = p$, $y'' = p\frac{dp}{dy}$ we obtain the Bernoulli equatior

$$p\frac{dp}{dy} + p^2 = 2e^{-y}.$$

By the substitution $p^2 = z$ it is reduced to the linear equation

$$\frac{dz}{dy} + 2z = 4e^{-y}$$

whose general solution is $z = 4e^{-y} + C_1 e^{-2y}$. Replacing z by $p^2 = y'^2$ we get

$$\frac{dy}{dx} = \pm\sqrt{4e^{-y} + C_1 e^{-2y}}.$$

Separating the variables and integrating we have

$$x + C_2 = \pm\frac{1}{2}\sqrt{4e^{y} + C_1}, \text{ whence } e^{y} + \widetilde{C}_1 = (x + C_2)^2,$$

where $\widetilde{C}_1 = C_1/4$.

This is the general integral of the given equation.

Example 6. Solve the equation $x^2 yy'' = (y - xy')^2$.

Solution. The equation is homogeneous in y, y', y''. The order of the equation is depressed by unity using the substitution $y = e^{\int z\,dx}$, where z is a new unknown function of x. We have

$$y' = ze^{\int z\,dx}, \quad y'' = (z' + z^2)\,e^{\int z\,dx}.$$

Substituting the expressions for y, y', y'' in the equation we get

$$x^2(z' + z^2)\,e^{2\int z\,dx} = (e^{\int z\,dx} - xz\,e^{\int z\,dx})^2.$$

We cancel $e^{2\int z\,dx}$:

$$x^2(z' + z^2) = (1 - xz)^2 \text{ or } x^2z' + 2xz = 1.$$

This equation is linear. Its left-hand side can be written as $(x^2z)' = 1$, whence

$$x^2z = x + C_1 \quad \text{or} \quad z = \frac{1}{x} + \frac{C_1}{x^2}.$$

We find the integral:

$$\int z\,dx = \int \left(\frac{1}{x} + \frac{C_1}{x^2}\right) dx = \ln|x| - \frac{C_1}{x} + \ln C_2.$$

The general solution of the given equation is

$$y = e^{\int z\,dx} = e^{\ln|x| - \frac{C_1}{x} + \ln C_2} \text{ or } y = C_2xe^{-\frac{C_1}{x}}.$$

Besides, the equation has an obvious solution, $y = 0$, which is obtained from the general solution when $C_2 = 0$.

Example 7. Solve the equation $x^3y'' = (y - xy')^2$.

Solution. We show that this equation is a generalized homogeneous equation. Taking x, y, y', y'' to be quantities of the first, mth, $(m - 1)$th and $(m - 2)$th degrees respectively and equating the degrees of all the terms we get

$$3 + (m - 2) = 2m, \tag{3}$$

whence $m = 1$. The solvability of equation (3) is the condition for the generalized homogeneity of the equation.

We make the substitution $x = e^t$, $y = ue^t$. Since

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\left(\frac{du}{dt}+u\right)e^t}{e^t}=\frac{du}{dt}+u,$$

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dt}\right)}{\frac{dx}{dt}}=\frac{\frac{d^2u}{dt^2}+\frac{du}{dt}}{e^t}=e^{-t}\left(\frac{d^2u}{dt^2}+\frac{du}{dt}\right)$$

the given equation, after cancelling the factor e^{2t}, takes the form

$$\frac{d^2u}{dt^2}+\frac{du}{dt}=\left(\frac{du}{dt}\right)^2.$$

Setting $\frac{du}{dt}=p$, $\frac{d^2u}{dt^2}=p\frac{dp}{du}$ we get $p\frac{dp}{du}+p=p^2$. Hence $p=0$ or $\frac{dp}{du}+1=p$. Integrating the second equation we find

$$p=1+C_1e^u \text{ or } \frac{du}{dt}=1+C_1e^u.$$

The general solution of this equation is

$$u=\ln\frac{e^t}{C_1e^t+C_2}.$$

Returning to the variables x and y we obtain the general solution of the given equation:

$$y=x\ln\frac{x}{C_1x+C_2}.$$

The case $p = 0$ gives $u = C$ or $y = Cx$, a particular solution obtained from the general one when $C_1 = e^{-C}$; $C_2 = 0$.

Remark. When solving the initial value problem for higher order equations it is advisable to determine the values of constants C_i in the process of solution rather than after finding the general solution of the equation. This accelerates the solution of the problem and, besides, it may turn out that integrating becomes much simpler when the constants C_i take particular numerical values while for arbitrary C_i it is difficult, or in general impossible, to integrate in elementary functions.

Example 8. Solve the initial value problem $y'' = 2y^3$; $y|_{x=0} = 1$, $y'|_{x=0} = 1$.

Solution. Setting $y' = p$ we get

$$p\frac{dp}{dy} = 2y^3$$

whence

$$p^2 = y^4 + C_1 \text{ or } \frac{dy}{dx} = \sqrt{y^4 + C_1}.$$

Separating the variables we find that

$$x + C_2 = \int (y^4 + C_1)^{-\frac{1}{2}}\, dy.$$

The right-hand side of the last equation contains an integral of differential binomial. Here $m = 0$, $n = 4$, $p = -1/2$, i.e. we have the nonintegrable case (see [13]).

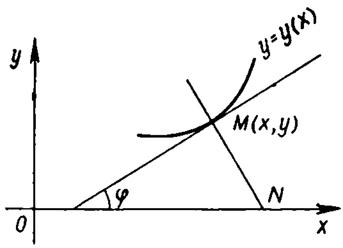

Fig. 24

Consequently, this integral cannot be expressed as a finite combination of elementary functions. However, if we use the initial conditions, then we get $C_1 = 0$. So $\frac{dy}{dx} = y^2$, whence, taking into account the initial conditions, we finally find that $y = \frac{1}{1-x}$.

Example 9. Find plane curves whose radius of curvature is proportional to the length of the normal.

Solution. Let $y = y(x)$ be the equation of the desired curve. Its radius of curvature is $R = \frac{(1+y'^2)^{3/2}}{|y''|}$. The length of the normal MN of the curve is (Fig. 24): $MN = |y|\sqrt{1 + y'^2}$.

The governing property of the curve is expressed by the differential equation

$$\frac{1+y'^2}{y''}=ky, \tag{4}$$

k being the proportionality coefficient capable of taking both positive and negative values. We rewrite equation (4) in the form

$$\frac{2y'y''}{1+y'^2}=\frac{2y'}{ky}.$$

Integrating we find

$$\ln(1+y'^2)=\frac{2}{k}(\ln|y|+\ln C_1) \text{ or } \frac{dy}{dx}=\sqrt{\left(\frac{y}{C_1}\right)^{2/k}-1}.$$

Separating the variables and integrating once again we get

$$x+C_2=\int\frac{dy}{\sqrt{\left(\frac{y}{C_1}\right)^{2/k}-1}}$$

which is the general integral of the original equation (4).

We shall consider some particular cases.

(1) $k=-1$. Then we have

$$x+C_2=\int\frac{y\,dy}{\sqrt{C_1^2-y^2}}$$

and, after integrating,

$$x+C_2=-\sqrt{C_1^2-y^2}.$$

Hence $(x+C_2)^2+y^2=C_1^2$. The desired curves are circles of arbitrary radii with centres on the Ox axis.

(2) $k=-2$. In this case we arrive at the equation

$$x+C_2=\int\sqrt{\frac{y}{C_1-y}}\,dy.$$

Setting $y=\frac{C_1}{2}(1-\cos t)$ we find that

$$\int\sqrt{\frac{y}{C_1-y}}\,dy=\frac{C_1}{2}(t-\sin t).$$

Thus the desired curves are determined in the parametric form by the equations

$$x+C_2=\frac{C_1}{2}(t-\sin t),\quad y=\frac{C_1}{2}(1-\cos t).$$

These are cycloids formed by the oscillation of circles of arbitrary radii along the Ox axis.

(3) $k = 1$. In this case we have

$$x+C_2=C_1\int\limits_{C_1}^{y}\frac{dy}{\sqrt{y^2-C_1^2}}=C_1\ln\frac{y+\sqrt{y^2-C_1^2}}{C_1},$$

whence

$$y+\sqrt{y^2+C_1^2}=C_1e^{\frac{x+C_2}{C_1}},\quad y-\sqrt{y^2-C_1^2}=C_1e^{-\frac{x+C_2}{C_1}}.$$

Adding the equations obtained we have

$$y=\frac{C_1}{2}\left(e^{\frac{x+C_2}{C_1}}+e^{-\frac{x+C_2}{C_1}}\right)=C_1\cosh\frac{x+C_2}{C_1};$$

these are catenaries.

(4) $k = 2$. Then we have

$$x+C_2=\int\frac{dy}{\sqrt{\frac{y}{C_1}-1}}\quad\text{or}\quad x+C_2=2C_1\sqrt{\frac{y}{C_1}-1}.$$

Hence $(x + C_2)^2 = 4C_1(y - C_1)$; these are parabolas whose axes are parallel to the Oy axis.

Integrate the following equations:

327. $y^{IV} = x$.

328. $y''' = x + \cos x$.

329. $y''(x+2)^5=1;\quad y(-1)=\frac{1}{12},\quad y'(-1)=-1/4$.

330. $y'' = xe^x,\quad y(0) = y'(0) = 0$.

331. $y'' = 2x\ln x$.

332. $xy'' = y'$.

333. $xy'' + y' = 0$.

334. $xy'' = (1 + 2x^2)\,y'$.

335. $xy'' = y' + x^3$.

336. $x\ln x\cdot y'' = y'$.

337. $xy = y'\ln\frac{y'}{x}$.

338. $2y''=\frac{y'}{x}+\frac{x^2}{y'};\quad y(1)=\frac{\sqrt{2}}{5},\quad y'(1)=\frac{\sqrt{2}}{2}$.

339. $y''' = \sqrt{1 - y''^2}$.

340. $xy''' - y'' = 0$.

341. $y'' = \sqrt{1 + y'^2}$.

342. $y'' = y'^2$.

343. $y'' = \sqrt{1 - y'^2}$.

344. $y'' = 1 + y'^2$.

345. $y'' = \sqrt{1 + y'}$.

346. $y'' = y' \ln y'$; $y(0) = 0$, $y'(0) = 1$.

347. $y'' + y' + 2 = 0$, $y(0) = 0$, $y'(0) = -2$.

348. $y'' = y'(1 + y')$.

349. $3y'' = (1 + y'^2)^{3/2}$.

350. $y''' + y''^2 = 0$.

351. $yy'' = y'^2$.

352. $y'' = 2yy'$; $y(0) = y'(0) = 1$.

353. $3y'y'' = 2y$; $y(0) = y'(0) = 1$.

354. $2y'' = 3y^2$; $y(-2) = 1$, $y'(-2) = -1$.

355. $yy'' + y'^2 = 0$.

356. $yy'' = y' + y'^2$.

357. $yy'' = 1 + y'^2$.

358. $2yy'' = 1 + y'^2$.

359. $y^3y'' = -1$; $y(1) = 1$, $y'(1) = 0$.

360. $yy'' - y'^2 = y^2y'$.

361. $y'' = e^{2y}$; $y(0) = 0$, $y'(0) = 1$.

362. $2yy'' - 3y'^2 = 4y^2$.

363. $y''' = 3yy'$; $y(0) = y'(0) = 1$, $y''(0) = 3/2$.

364. Find the plane curves whose radius of curvature is proportional to the cube of the normal.

365. Determine the form of the equilibrium of an inextensible thread which is acted on by a load in such a way that the loading per each unit of the plan is the same (the chains of a chain bridge).

366. Find the time required for a body to fall to the Earth from the height of 400,000 km (an approximate distance of the Moon from the centre of the Earth), if the height is measured from the centre of the Earth and if the radius of the Earth is about 6,400 km.

367. Find the law of motion of a particle of mass m along a straight line OA under a repulsive force inversely proportional to the third power of the distance of a point $x = OM$ from the fixed centre O.

368. A body of mass m falls from some height with velocity v. While falling it overcomes the resistance proportional to the square of the velocity. Find the law of motion of the falling body.

369. Find the curve which passes through the origin of coordinates and is such that the area of the triangle formed by the tangent to the curve at some point, the ordinate of that point and the Ox axis is proportional to the area of the curvilinear trapezoid formed by the curve, the Ox axis and the ordinate of the point.

370. Determine the curve whose radius of curvature is equal to a constant.

15. Linear differential equations of the nth order

15.1. Linear independence of functions. The Wronskian determinant. The Gramian. Let $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ be a finite system of n functions defined on the interval (a, b). The functions $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ are said to be *linearly dependent* in the interval (a, b) if there exist constants α_1, α_2, ..., α_n, not all equal to zero, such that for all values of x in this interval the identity

$$\alpha_1 y_1(x) + \alpha_2 y_2(x) + \ldots + \alpha_n y_n(x) \equiv 0$$

is valid. If, however, the identity holds only for $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, then the functions $y_1(x), y_2(x), \ldots, y_n(x)$ are said to be *linearly independent* in the interval (a, b).

Example 1. Show that the system of functions 1, x, x^2, x^3 is linearly independent in the interval $(-\infty, +\infty)$.

Solution. The equation $\alpha_1 \cdot 1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 = 0$ may indeed hold for all $x \in (-\infty, +\infty)$ only provided that

$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. If, however, at least one of these numbers is nonzero, then the left-hand side of the equation will contain a polynomial of degree not higher than 3 and it may vanish for at most three values of x in the given interval.

Example 2. Show that the system of functions e^{k_1x}, e^{k_2x}, e^{k_3x}, where k_1, k_2, k_3 are in pairs different, is linearly independent in the interval $-\infty < x < +\infty$.

Solution. Suppose the converse, i.e. that the given system is linearly dependent in this interval. Then

$$\alpha_1 e^{k_1x} + \alpha_2 e^{k_2x} + \alpha_3 e^{k_3x} \equiv 0 \tag{1}$$

on the interval $(-\infty, +\infty)$, at least one of the numbers α_1, α_2, α_3 being nonzero, for example $\alpha_3 \neq 0$. Dividing both sides of the identity (1) by e^{k_1x} we have

$$\alpha_1 + \alpha_2 e^{(k_2-k_1)x} + \alpha_3 e^{(k_3-k_1)x} \equiv 0.$$

Differentiating the identity we get

$$\alpha_2 (k_2 - k_1) e^{(k_2-k_1)x} + \alpha_3 (k_3 - k_1) e^{(k_3-k_1)x} \equiv 0. \tag{2}$$

We divide both sides of identity (2) by $e^{(k_2-k_1)x}$:

$$\alpha_2 (k_2 - k_1) + \alpha_3 (k_3 - k_1) e^{(k_3-k_2)x} \equiv 0. \tag{3}$$

Differentiating (3) we get

$$\alpha_3 (k_3 - k_1)(k_3 - k_2) e^{(k_3-k_2)x} \equiv 0$$

which is impossible, since $a_3 \neq 0$ by the supposition, $k_3 \neq k_1$, $k_3 \neq k_2$ according to the condition, and $e^{(k_3-k_2)x} \neq 0$.

Our supposition that the given system of functions is linearly dependent has led to a contradiction, hence this system of functions is linearly independent in the interval $(-\infty, +\infty)$, i.e. identity (1) will hold only for $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Example 3. Show that the system of functions $e^{\alpha x}\sin\beta x$, $e^{\alpha x}\cos\beta x$, where $\beta \neq 0$, is linearly independent in the interval $-\infty < x < +\infty$.

Solution. We determine the values of α_1 and α_2 for which the identity

$$\alpha_1 e^{\alpha x}\sin\beta x + \alpha_2 e^{\alpha x}\cos\beta x \equiv 0 \tag{4}$$

will hold. Dividing both of its sides by $e^{\alpha x} \neq 0$ we have

$$\alpha_1 \sin\beta x + \alpha_2 \cos\beta x \equiv 0. \tag{5}$$

We substitute in (5) the value of $x = 0$ to get $\alpha_2 = 0$ and hence $\alpha_1 \sin \beta x \equiv 0$; but the function $\sin \beta x$ is not identically equal to zero, therefore $\alpha_1 = 0$. Identity (5), and hence, (4) hold only when $\alpha_1 = \alpha_2 = 0$, i.e. the given functions are linearly independent in the interval $-\infty < x < +\infty$.

Remark. Incidentally we have proved the linear independence of the trigonometric functions $\sin \beta x$, $\cos \beta x$.

Example 4. Prove that the functions

$$\sin x, \ \sin\left(x+\frac{\pi}{8}\right), \quad \sin\left(x-\frac{\pi}{8}\right) \tag{6}$$

are linearly dependent in the interval $(-\infty, +\infty)$.

Solution. We show that there are numbers α_1, α_2, α_3 not all being equal to zero, such that in the interval $-\infty < x < +\infty$ the identity

$$\alpha_1 \sin x = \alpha_2 \sin\left(x+\frac{\pi}{8}\right) + \alpha_3 \sin\left(x-\frac{\pi}{8}\right) \equiv 0 \tag{7}$$

is valid. We assume identity (7) to hold; we set, for example, $x = 0$, $x = \pi/4$, $x = \pi/2$. Then we shall obtain a homogeneous system of three equations in three unknowns α_1, α_2, α_3:

$$\begin{cases} \alpha_2 \sin\dfrac{\pi}{8} - \alpha_3 \sin\dfrac{\pi}{8} = 0, \\ \alpha_1 \dfrac{1}{\sqrt{2}} + \alpha_2 \sin\dfrac{3\pi}{8} + \alpha_3 \sin\dfrac{\pi}{8} = 0, \\ \alpha_1 + \alpha_2 \sin\dfrac{5\pi}{8} + \alpha_3 \sin\dfrac{3\pi}{8} = 0. \end{cases} \tag{8}$$

The determinant of this system is

$$\Delta = \begin{vmatrix} 0 & \sin\dfrac{\pi}{8} & -\sin\dfrac{\pi}{8} \\ \dfrac{1}{\sqrt{2}} & \sin\dfrac{3\pi}{8} & \sin\dfrac{\pi}{8} \\ 1 & \sin\dfrac{5\pi}{8} & \sin\dfrac{3\pi}{8} \end{vmatrix} = 0.$$

Consequently, the homogeneous system (8) has nonzero solutions, i.e. there exist numbers α_1, α_2, α_3 among which there is at least one which is nonzero. To find such a triple of numbers α_1, α_2, α_3, take, for example, the first two equa-

tions of system (8):

$$\alpha_2 \sin\frac{\pi}{8} - \alpha_3 \sin\frac{\pi}{8} = 0,$$

$$\frac{\alpha_1}{\sqrt{1}} + \alpha_2 \sin\frac{3}{8}\pi + \alpha_3 \sin\frac{\pi}{8} = 0.$$

The first equation gives $\alpha_2 = \alpha_3$, the second gives $\alpha_1 = -2\cos\frac{\pi}{8} \times \alpha_3$. Setting $\alpha_3 = 1$ we obtain a nonzero solution of system (8):

$$\alpha_1 = -2\cos\frac{\pi}{8}, \quad \alpha_2 = 1, \quad \alpha_3 = 1.$$

We now show that with all these values of α_1, α_2, α_3 identity (7) will hold for all $x \in (-\infty, +\infty)$*. We have

$$\alpha_1 \sin x + \alpha_2 \sin\left(x + \frac{\pi}{8}\right) + \alpha_3 \sin\left(x - \frac{\pi}{8}\right) \equiv$$

$$\equiv -2\cos\frac{\pi}{8}\sin x + 2\sin x \cos\frac{\pi}{8} \equiv 0,$$

whatever the value of x. Consequently, the system of functions (6) is linearly dependent in the interval $-\infty < x < +\infty$.

Remark. For the case of two functions a simpler criterion for linear independence can be given. Namely, functions $\varphi_1(x)$ and $\varphi_2(x)$ will be linearly independent in the interval (a, b) if their ratio is not identically equal to a constant $\left(\frac{\varphi_1(x)}{\varphi_2(x)} \not\equiv \text{const}\right)$ in this interval; if, however, $\frac{\varphi_1(x)}{\varphi_2(x)} \equiv \text{const}$, then the functions will be linearly dependent.

Example 5. The functions $\tan x$ and $\cot x$ are linearly independent in the interval $0 < x < \frac{\pi}{2}$, since their ratio $\frac{\tan x}{\cot x} = \tan^2 x \not\equiv \text{const}$ in this interval.

Example 6. The functions $\sin 2x$ and $\sin x \cos x$ are linearly dependent in the interval $-\infty < x < +\infty$, since

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* The linear dependence of $\sin x$, $\sin\left(x + \frac{\pi}{8}\right)$, $\sin\left(x - \frac{\pi}{8}\right)$ can be established if we notice that $\sin\left(x + \frac{\pi}{8}\right) + \sin\left(x - \frac{\pi}{8}\right) = 2\cos\frac{\pi}{8}\sin x$ or $\sin\left(x + \frac{\pi}{8}\right) + \sin\left(x - \frac{\pi}{8}\right) - 2\cos\frac{\pi}{8}\sin x \equiv 0$.

their ratio $\frac{\sin 2x}{\sin x \cos x} \equiv \frac{2 \sin x \cos x}{\sin x \cos x} \equiv 2 = \text{const}$ in this interval (at the points of discontinuity of the function $\frac{\sin 2x}{\sin x \cos x}$ we extend this ratio by the continuity).

Let n functions $y_1(x), y_2(x), \ldots, y_n(x)$ have derivatives of the $(n-1)$th order. The determinant

$$W[y_1, y_2, \ldots, y_n] = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix}$$

is called the *Wronskian determinant* for these functions. It is in general a function of x defined in some interval.

Example 7. Find the Wronskian determinant for the functions $y_1(x) = e^{k_1 x}$, $y_2(x) = e^{k_2 x}$, $y_3(x) = e^{k_3 x}$.

Solution. We have

$$W[y_1, y_2, y_3] = \begin{vmatrix} e^{k_1 x} & e^{k_2 x} & e^{k_3 x} \\ k_1 e^{k_1 x} & k_2 e^{k_2 x} & k_3 e^{k_3 x} \\ k_1^2 e^{k_1 x} & k_2^2 e^{k_2 x} & k_3^2 e^{k_3 x} \end{vmatrix}$$
$$= e^{(k_1+k_2+k_3)x}(k_2-k_1)(k_3-k_1)(k_3-k_2).$$

Example 8. Find the Wronskian determinant for the functions: $y_1(x) = \sin x$, $y_2(x) = \sin\left(x + \frac{\pi}{8}\right)$, $y_3(x) = \sin\left(x - \frac{\pi}{8}\right)$.

Solution. We have

$$W[y_1, y_2, y_3]$$
$$= \begin{vmatrix} \sin x & \sin\left(x+\frac{\pi}{8}\right) & \sin\left(x-\frac{\pi}{8}\right) \\ \cos x & \cos\left(x+\frac{\pi}{8}\right) & \cos\left(x-\frac{\pi}{8}\right) \\ -\sin x & -\sin\left(x+\frac{\pi}{8}\right) & -\sin\left(x-\frac{\pi}{8}\right) \end{vmatrix} = 0$$

since the first and the last row of the determinant are proportional.

Theorem. *If a system of linear functions $y_1(x), y_2(x), \ldots, y_n(x)$ is linearly dependent in the interval $[a, b]$ then its Wronskian is identically equal to zero in this interval.*

Thus, for example, the system of functions $\sin x$, $\sin\left(x+\frac{\pi}{8}\right)$, $\sin\left(x-\frac{\pi}{8}\right)$ is linearly dependent in the interval $(-\infty, +\infty)$ and the Wronskian of these functions is zero everywhere in this interval (see Examples 4 and 8).

This theorem gives the necessary condition for the linear dependence of a system of functions. The converse is not

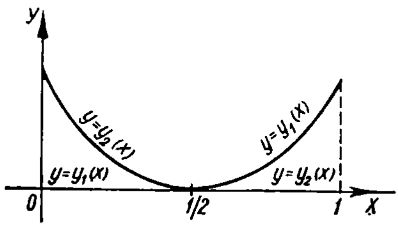

Fig. 25

true, i.e. the Wronskian may identically vanish even when the given functions form a linearly independent system in some interval.

Example 9. Consider the two functions:

$$y_1(x)=\begin{cases} 0 \text{ if } 0\leqslant x\leqslant\frac{1}{2}, \\ \left(x-\frac{1}{2}\right)^2 \text{ if } \frac{1}{2}<x\leqslant 1; \end{cases}$$

$$y_2(x)=\begin{cases} \left(x-\frac{1}{2}\right)^2 \text{ if } 0\leqslant x\leqslant\frac{1}{2}, \\ 0 \text{ if } \frac{1}{2}<x\leqslant 1. \end{cases}$$

Their graphs are of the form indicated in Fig. 25.

This system of functions is linearly independent since the identity $\alpha_1 y_1(x)+\alpha_2 y_2(x)\equiv 0$ holds only for $\alpha_1 = \alpha_2 = 0$. Indeed, considering it in the interval $\left[0, \frac{1}{2}\right]$, we get $\alpha_2 y_2(x)\equiv 0$, whence $\alpha_2=0$, since $y_2(x)\not\equiv 0$; in the interval $\left[\frac{1}{2}, 1\right]$, however, we have $\alpha_1 y_1(x)\equiv 0$, whence $\alpha_1=0$, since $y_1(x)\not\equiv 0$ in this interval.

We find the Wronskian $W[y_1, y_2]$ of the system. In the interval $\left[0, \frac{1}{2}\right]$

$$W[y_1, y_2] = \begin{vmatrix} 0 & \left(x-\frac{1}{2}\right)^2 \\ 0 & 2\left(x-\frac{1}{2}\right) \end{vmatrix} = 0,$$

in the interval $\left[\frac{1}{2}, 1\right]$

$$W[y_1, y_2] = \begin{vmatrix} \left(x-\frac{1}{2}\right)^2 & 0 \\ 2\left(x-\frac{1}{2}\right) & 0 \end{vmatrix} = 0.$$

Thus the Wronskian $W[y_1, y_2] \equiv 0$ in the interval $[0, 1]$.

Let $y_1(x), y_2(x), \ldots, y_n(x)$ be a system of functions given in the interval $[a, b]$. We set

$$(y_i, y_j) = \int_a^b y_i(x)\, y_j(x)\, dx, \quad i, j = 1, 2, \ldots, n.$$

The determinant

$$\Gamma(y_1, y_2, \ldots, y_n) = \begin{vmatrix} (y_1, y_1) & (y_1, y_2) & \ldots & (y_1, y_n) \\ (y_2, y_1) & (y_2, y_2) & \ldots & (y_2, y_n) \\ (y_n, y_1) & (y_n, y_2) & \ldots & (y_n, y_n) \end{vmatrix}$$

is called *the Gramian* of the system of functions $\{y_k(x)\}$.

Theorem. *For a system of functions* $y_1(x), y_2(x), \ldots, y_n(x)$ *to be linearly dependent it is necessary and sufficient that its Gramian should be zero.*

Example 10. Show that the functions $y_1 = x$, $y_2 = 2x$ are linearly dependent in the interval $[0, 1]$.

Solution. We have

$$(y_1, y_1) = \int_0^1 x^2\, dx = \frac{1}{3}; \quad (y_1, y_2) = (y_2, y_1)$$

$$= \int_0^1 2x^2\, dx = \frac{2}{3}; \quad (y_2, y_2) = \int_0^1 4x^2\, dx = \frac{4}{3};$$

$$\Gamma(y_1, y_2) = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{vmatrix} = 0,$$

consequently, the functions $y_1(x)$ and $y_2(x)$ are linearly dependent.

Examine whether the given functions are linearly independent in their domain of definition.

371. $4, x$.

372. $1, 2, x, x^2$.

373. $x, 2x, x^2$.

374. $e^x, xe^x, x^2 e^x$.

375. $\sin x, \cos x, \cos 2x$.

376. $1, \sin x, \cos 2x$.

377. $5, \cos^2 x, \sin^2 x$.

378. $\cos x, \cos(x+1), \cos(x-2)$.

379. $1, \sin 2x, (\sin x - \cos x)^2$.

380. $x, a^{\log_a x}$ $(x>0)$.

381. $\log_a x, \log_a x^2$ $(x>0)$.

382. $1, \operatorname{arc\,sin} x, \operatorname{arc\,cos} x$.

383. $5, \operatorname{arc\,tan} x, \operatorname{arc\,cot} x$.

384. $2\pi, \operatorname{arc\,tan}\frac{x}{2\pi}, \operatorname{arc\,cot}\frac{x}{2\pi}$.

385. $e^{-\frac{ax^2}{2}}, \quad e^{-\frac{ax^2}{2}}\int\limits_0^x e^{\frac{at^2}{2}}\,dt$.

386. $x, \; x\int\limits_{x_0}^{x}\frac{e^t}{t^2}\,dt \; (x_0>0)$.

387. Show that the system of functions

$$y_1(x), y_1(x), y_2(x), \ldots, y_{n-1}(x)$$

defined in the interval (a, b) is linearly dependent in (a, b).

388. Show that if the system of functions

$$y_1(x), y_2(x), \ldots, y_n(x)$$

is linearly independent in the interval (a, b), then any subsystem of this system of functions is also linearly independent in (a, b).

In the problems below find the Wronskian for the indicated systems of functions:

389. $1, x$.

390. $x, \frac{1}{x}$.

391. $1, 2, x^2$.

392. e^{-x}, xe^{-x}.

393. $e^x, 2e^x, e^{-x}$.

394. $2, \cos x, \cos 2x$.

395. $\sin x, \sin\left(x+\frac{\pi}{4}\right)$.

396. $\arccos\frac{x}{\pi}, \quad \arcsin\frac{x}{\pi}$.

397. $\pi, \ \arcsin x, \ \arccos x$.

398. $4, \sin^2 x, \cos 2x$.

399. $x, \ln x$.

400. $\frac{1}{x}, \ e^{\frac{1}{x}}$.

401. $e^x \sin x, e^x \cos x$.

402. $e^{-3x} \sin 2x, e^{-3x} \cos 2x$.

403. $\cos x, \sin x$.

404. $\sin\left(\frac{\pi}{4}-x\right), \cos\left(\frac{\pi}{4}-x\right)$.

In the following problems show that the given functions are linearly independent and that their Wronskian is identically equal to zero and construct the graphs of these functions:

405. $y_1(x)=\begin{cases} x^2 & \text{if} \quad -1\leqslant x\leqslant 0; \\ 0 & \text{if} \quad -0< x\leqslant 1. \end{cases}$

$y_2(x)=\begin{cases} 0 & \text{if} \quad -1\leqslant x\leqslant 0; \\ x^2 & \text{if} \quad 0< x\leqslant 1. \end{cases}$

406. $y_1(x)=\begin{cases} 0 & \text{if} \quad 0\leqslant x\leqslant 2; \\ (x-2)^2 & \text{if} \quad 2< x\leqslant 4. \end{cases}$

$y_2(x)=\begin{cases} (x-2)^2 & \text{if} \quad 0\leqslant x\leqslant 2; \\ 0 & \text{if} \quad 2< x\leqslant 4. \end{cases}$

407. $y_1(x)=\begin{cases} x^3 & \text{if} \quad -2\leqslant x\leqslant 0; \\ 0 & \text{if} \quad 0< x\leqslant 1. \end{cases}$

$y_2(x)=\begin{cases} 0 & \text{if} \quad -2\leqslant x\leqslant 0; \\ x^2 & \text{if} \quad 0< x\leqslant 1. \end{cases}$

408. $y_1(x) = x^2$, $y_2(x) = x\,|\,x\,|$; $-1 \leqslant x \leqslant 1$.

409. Using the Gramian show that the systems of functions in Problems 373, 377, and 379 are linearly dependent in $[-\pi, \pi]$.

15.2. Homogeneous linear equations with constant coefficients. Consider the differential equation

$$a_0y^{(n)} + a_1y^{(n-1)} + \dots + a_ny = 0, \tag{9}$$

where $a_0, a_1, \dots, a_n$ are real constants, $a_0 \neq 0$.

To find the general solution of equation (9) we proceed as follows.

1. We set up for equation (9) the characteristic equation

$$a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_n = 0. \tag{10}$$

2. We find the roots $\lambda_1, \lambda_2, \dots, \lambda_n$ of the characteristic equation (10).

3. According to the nature of the roots of the characteristic equation (10) we write out linearly independent particular solutions of the differential equation (9) taking into account the fact that

(a) corresponding to each real single root λ of the characteristic equation (10) is a particular solution $e^{\lambda x}$ of the differential equation (9);

(b) corresponding to each single pair of complex conjugate roots $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ of the characteristic equation (10) are two linearly independent particular solutions $e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$ of the differential equation (9);

(c) corresponding to each real root λ of multiplicity s of the characteristic equation (10) are s linearly independent particular solutions $e^{\lambda x}$, $xe^{\lambda x}$, $x^2e^{\lambda x}$, $\dots$, $x^{s-1}e^{\lambda x}$ of the differential equation (9);

(d) corresponding to each pair of complex conjugate roots $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ of multiplicity s are $2s$ linearly independent particular solutions of the differential equation (9)

$$e^{\alpha x}\cos\beta x, \quad xe^{\alpha x}\cos\beta x, \ \dots, \ x^{s-}\ e^{\alpha x}\cos\beta x,$$

$$e^{\alpha x}\sin\beta x, \quad xe^{\alpha x}\sin\beta x, \ \dots, \ x^{s-}\ e^{\alpha x}\sin\beta x.$$

The number of particular solutions of the differential equation (9) thus constructed is equal to the order of the equation.

All the solutions constructed are linearly independent in the aggregate and make up the fundamental system of solutions of the differential equation (9).

Example 1. Find the general solution of the equation

$$y''' - 2y'' - 3y' = 0.$$

Solution. We set up the characteristic equation $\lambda^3 - 2\lambda^2 - 3\lambda = 0$.
We find its roots: $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = 3$. Since they are real and distinct, the general solution is of the form

$$y_{g.h} = C_1 + C_2 e^{-x} + C_3 e^{3x}.$$

Example 2. Find the general solution of the equation

$$y''' + 2y'' + y' = 0.$$

Solution. The characteristic equation is of the form

$$\lambda^3 + 2\lambda^2 + \lambda = 0.$$

Hence $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 0$. The roots are real, one of them, namely $\lambda_1 = -1$, being double, therefore the general solution is of the form

$$y_{g.h} = C_1 e^{-x} + C_2 x e^{-x} + C_3.$$

Example 3. Find the general solution of the equation $y''' + 4y'' + 13y' = 0$.

Solution. The characteristic equation

$$\lambda^3 + 4\lambda^2 + 13\lambda = 0$$

has the roots $\lambda_1 = 0$, $\lambda_2 = -2 - 3i$, $\lambda_3 = -2 + 3i$.

The general solution s

$$y_{g.h} = C_1 + C_2 e^{-2x} \cos 3x + C_3 e^{-2x} \sin 3x.$$

Example 4. Find the general solution of the equation

$$y^{V} - 2y^{IV} + 2y''' - 4y'' + y' - 2y = 0.$$

Solution. The characteristic equation

$$\lambda^5 - 2\lambda^4 + 2\lambda^3 - 4\lambda^2 + \lambda - 2 = 0$$

or

$$(\lambda - 2)(\lambda^2 + 1)^2 = 0$$

has the roots $\lambda = 2$, a single one, and $\lambda = \pm i$, a pair of double imaginary roots. The general solution is

$$y_{g.h} = C_1 e^{2x} + (C_2 + C_3 x) \cos x + (C_4 + C_5 x) \sin x.$$

Example 5. Solve the equation

$$y^{IV} + 4y''' + 8y'' + 8y' + 4y = 0.$$

Solution. We set up the characteristic equation

$$\lambda^4 + 4\lambda^3 + 8\lambda^2 + 8\lambda + 4 = 0 \text{ or } (\lambda^2 + 2\lambda + 2)^2 = 0.$$

It has double complex roots $\lambda_1 = \lambda_2 = -1 - i$, $\lambda_3 = \lambda_4 = -1 + i$ and, consequently, the general solution is of the form

$$y_{g.h} = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x + C_3 x e^{-x} \cos x + C_4 x e^{-x} \sin x$$

or

$$y_{g.h} = e^{-x} (C_1 + C_3 x) \cos x + e^{-x} (C_2 + C_4 x) \sin x.$$

Set up homogeneous linear differential equations if their characteristic equations are known:

410. $9\lambda^2 - 6\lambda + 1 = 0$.

411. $\lambda^2 + 3\lambda + 2 = 0$.

412. $2\lambda^2 - 3\lambda - 5 = 0$.

413. $\lambda (\lambda + 1)(\lambda + 2) = 0$.

414. $(\lambda^2 + 1)^2 = 0$.

415. $\lambda^3 = 0$.

Set up homogeneous linear differential equations, if the roots of the characteristic equations are known, and write their general solutions:

416. $\lambda_1 = 1$, $\lambda_2 = 2$.

417. $\lambda_1 = 1$, $\lambda_2 = 1$.

418. $\lambda_1 = 3 - 2i$, $\lambda_2 = 3 + 2i$

419. $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$.

Set up homogeneous linear differential equations, given their fundamental systems of solutions:

420. e^{-x}, e^x.

421. 1, e^x.

422. e^{-2x}, xe^{-2x}.

423. $\sin 3x$, $\cos 3x$.

424. 1, x.

425. e^x, e^{2x}, e^{3x}.

426. e^x, xe^x, x^2e^x.

427. e^x, xe^x, e^{2x}.

428. 1, x, e^x.

429. 1, $\sin x$, $\cos x$.

430. e^{2x}, $\sin x$, $\cos x$.

431. 1, $e^{-x} \sin x$, $e^{-x} \cos x$.

Integrate the following equations and solve the initial value problem, where indicated.

432. $y'' - y = 0.$

433. $3y'' - 2y' - 8y = 0.$

434. $y''' - 3y'' + 3y' - y = 0,$

$y(0) = 1, \quad y'(0) = 2, \; y''(0) = 3.$

435. $y'' + 2y' + y = 0.$

436. $y'' - 4y' + 3y = 0.$

$y(0) = 6, \quad y'(0) = 10.$

437. $y''' + 6y'' + 11y' + 6y = 0.$

438. $y'' - 2y' - 2y = 0.$

439. $y^{\mathrm{VI}} + 2y^{\mathrm{V}} + y^{\mathrm{IV}} = 0.$

440. $4y'' - 8y' + 5y = 0.$

441. $y''' - 8y = 0.$

442. $y^{\mathrm{IV}} + 4y''' + 10y'' + 12y' + 5y = 0.$

443. $y'' - 2y' + 2y = 0, \; y(0) = 0, \; y'(0) = 1.$

444. $y'' - 2y' + 3y = 0, \; y(0) = 1, \; y'(0) = 3.$

445. $y^{\mathrm{IV}} + 2y''' + 4y'' - 2y' - 5y = 0.$

446. $y^{\mathrm{IV}} + 4y^{\mathrm{IV}} + 5y''' - 6y' - 4y = 0.$

447. $y''' + 2y'' - y' - 2y = 0.$

448. $y''' - 2y'' + 2y' = 0.$

449. $y^{\mathrm{IV}} - y = 0.$

450. $y^{\mathrm{X}} = 0.$

451. $y''' - 3y' - 2y = 0.$

452. $2y''' - 3y'' + y' = 0.$

453. $y''' + y'' = 0; \; y(0) = 1, \; y'(0) = 0, \; y''(0) = 1.$

15.3. Nonhomogeneous linear equations with constant coefficients. A. *The trial and error method.* Let

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = f(x) \tag{11}$$

be a differential equation with real constant coefficients $a_0, a_1, a_2, \ldots, a_n$.

Theorem. *The general solution of the nonhomogeneous equation* (11) *is equal to the sum of the general solution of the corresponding homogeneous equation and some particular solution of the nonhomogeneous equation.*

The determination of the general solution of the corresponding homogeneous equation is effected by the rules presented in Sec. 15.2. Thus the problem of integrating equation (11) reduces to the determination of a particular solution $y_{\text{p. n}}$ of the nonhomogeneous equation. In the general case equation (11) can be integrated using the method of variation of arbitrary parameters (see Sec. 15.5 below). For the right-hand sides of *special form* the particular solution is easier to find by the so-called *trial and error method.* The general form of the right-hand side $f(x)$ of equation (11) allowing the use of the error method is as follows:

$$f(x) = e^{\alpha x}[P_l(x)\cos\beta x + Q_m(x)\sin\beta x],$$

$P_l(x)$ and $Q_m(x)$ being polynomials of degree l and m respectively. In this case a particular solution $y_{\text{p. n}}$ of equation (11) is sought in the form

$$y_{\text{p. n}} = x^s e^{\alpha x}[\widetilde{P}_k(x)\cos\beta x + \widetilde{Q}_k(x)\sin\beta x],$$

where $k = \max(m, l)$, $\widetilde{P}_k(x)$ and $\widetilde{Q}_k(x)$ are polynomials of the kth degree of the general form with undetermined coefficients and s is the multiplicity of the root $\lambda = \alpha + i\beta$ of the characteristic equation (if $\alpha \pm i\beta$ is not a root of the characteristic equation, then $s = 0$).

Example 1. Find the general solution of the equation $y''' - y'' + y' - y = x^2 + x$.

Solution. The characteristic equation $\lambda^3 - \lambda^2 + \lambda - 1 = 0$ has distinct roots: $\lambda_1 = 1$, $\lambda_2 = -i$, $\lambda_3 = i$, therefore the general solution of the corresponding homogeneous solution $y_{\text{g. h}}$ is

$$y_{\text{g. h}} = C_1 e^x + C_2\cos x + C_3\sin x.$$

Since the number zero is not a root of the characteristic equation, the particular solution of the given equation $y_{\text{p. n}}$ must be sought in the form (see Table 1, case I (1)):

$$y_{\text{p. n}} = A_1x^2 + A_2x + A_3,$$

where A_1, A_2, A_3 are as yet unknown coefficients subject to determination. Substituting the expression for $y_{\text{p. n}}$ in

Table 1. A summary of the forms of particular solutions for various right-hand sides *

No.	Right-hand side of differential equation	Roots of characteristic equation	Forms of particular solutions
I	$P_m(x)$	1. Number 0 is not a root of characteristic equation	$\widetilde{P}_m(x)$
		2. Number 0 is a root of characteristic equation of multiplicity s	$x^s\widetilde{P}_m(x)$
II	$P_m(x)\,e^{*}$	1. Number α is not a root of characteristic equation	$\widetilde{P}_m(x)\,e^{\alpha x}$
		2. Number α is a root of characteristic equation of multiplicity s	$x^s\widetilde{P}_m(x)\,e^{\alpha x}$
III	$P_n(x)\cos\beta x+$ $+Q_m(x)\sin\beta x$	1. Numbers $\pm i\beta$ are not roots of characteristic equation	$\widetilde{P}_k(x)\cos\beta x+$ $+\widetilde{Q}_k(x)\sin\beta x$
		2. Numbers $\pm i\beta$ are roots of characteristic equation of multiplicity s	$x^s(\widetilde{P}_k(x)\cos\beta x+$ $+\widetilde{Q}_k(x)\sin\beta x)$
IV	$e^{\alpha x}[P_n(x)\cos\beta x+$ $+Q_m(x)\sin\beta x]$	1. Numbers $\alpha\pm i\beta$ are not roots of characteristic equation	$(\widetilde{P}_k(x)\cos\beta x+$ $+\widetilde{Q}_k(x)\sin\beta x)\,e^{\alpha x}$
		2. Numbers $\alpha\pm i\beta$ are roots of characteristic equation of multiplicity s	$x^s(\widetilde{P}_k(x)\cos\beta x+$ $+\widetilde{Q}_k(x)\sin\beta x)\,e^{\alpha x}$

* The first three forms of right-hand sides are particular cases of form IV.

the given equation we get

$$-A_1x^2 + (2A_1 - A_2)\,x + (A_2 - 2A_1 - A_3) = x^2 + x,$$

whence

$$\begin{cases} A_1 = -1, \\ 2A_1 - A_2 = 1, \\ A_2 - 2A_1 - A_3 = 0. \end{cases}$$

Solving this system we find $A_1 = -1$, $A_2 = -3$, $A_3 = -1$, consequently, the particular solution is

$$y_{\text{p. n}} = -x^2 - 3x - 1$$

and the general solution $y_{\text{g. n}}$ of the given equation is of the form

$$y_{\text{g. n}} = C_1e^x + C_2\cos x + C_3\sin x - x^2 - 3x - 1.$$

Example 2. Find the general solution of the equation $y''' - y'' = 12x^2 + 6x$.

Solution. The characteristic equation $\lambda^3 - \lambda^2 = 0$ has the roots $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1$, and therefore the general solution of the corresponding homogeneous equation is

$$y_{\text{g. h}} = C_1 + C_2x + C_3e^x.$$

Since the number 0 is a double root of the characteristic equation, the particular solution must be sought in the form (see Table 1, case I (2))

$$y_{\text{p. n}} = x^2\,(A_1x^2 + A_2x + A_3) = A_1x^4 + A_2x^3 + A_3x^2.$$

Substituting the expression for $y_{\text{p. n}}$ in the given equation we have

$$-12\,A_1x^2 + (24A_1 - 6A_2)\,x + (6A_2 - 2A_3) = 12x^2 + 6x$$

whence

$$\begin{cases} -12A_1 = 12, \\ 24A_1 - 6A_2 = 6, \\ 6A_2 - 2A_3 = 0. \end{cases}$$

This system has the solution: $A_1 = -1$, $A_2 = -5$, $A_3 = -15$, and hence

$$y_{\text{p. n}} = -x^4 - 5x^3 - 15x^2.$$

The general solution of the given equation is

$$y_{\text{g. n}} = C_1 + C_2x + C_3e^x - x^4 - 5x^3 - 15x^2.$$

Example 3. Find the general solution of the equation $y'' + y' = 4x^2e^x$.

Solution. The characteristic equation $\lambda^2 + \lambda = 0$ has the roots $\lambda_1 = 0$, $\lambda_2 = -1$. Hence the general solution $y_{\text{g. h}}$ of the corresponding homogeneous equation is

$$y_{\text{g. h}} = C_1 + C_2e^{-x}.$$

Since $\alpha = 1$ is not a root of the characteristic equation, the particular solution $y_{\text{p. n}}$ of the nonhomogeneous equation is sought in the form (see Table 1, case II (1))

$$y_{\text{p. n}} = (A_1x^2 + A_2x + A_3)\, e^x.$$

Substituting it in the original equation and cancelling e^x from both sides of the equation we have

$$2A_1x^2 + (6A_1 + 2A_2)\, x + 2A_1 + 3A_2 + 2A_3 = 4x^2.$$

Equating the coefficients of the equal powers of x on the left and right of the equation we obtain a linear system of equations for finding the coefficients A_1, A_2, A_3:

$$\begin{cases} 2A_1 = 4, \\ 6A_1 + 2A_2 = 0, \\ 2A_1 + 3A_2 + 2A_3 = 0, \end{cases}$$

solving which we find $A_1 = 2$, $A_2 = -6$, $A_3 = 7$, so that

$$y_{\text{p. n}} = (2x^2 - 6x + 7)\, e^x.$$

The general solution of the given equation is

$$y(x) = C_1 + C_2e^{-x} + (2x^2 - 6x + 7)\, e^x.$$

Example 4. Find the general solution of the equation

$$y'' + 10y' + 25y = 4e^{-5x}.$$

Solution. The characteristic equation $\lambda^2 + 10\lambda + 25 = 0$ has a double root, $\lambda_1 = \lambda_2 = -5$, therefore

$$y_{\text{g. h}} = (C_1 + C_2x)\, e^{-5x}.$$

Since $\alpha = -5$ is a root of the characteristic equation of multiplicity $s = 2$, the particular solution $y_{\text{p. n}}$ of the nonhomogeneous equation is sought in the form (see Table 1, case II (2))

$$y_{\text{p. n}} = Bx^2e^{-5x}; \text{ then}$$

$$y'_{\text{p. n}} = B\,(2x - 5x^2)\, e^{-5x},$$

$$y''_{\text{p. n}} = B\,(2 - 20x + 25x^2)\, e^{-5x}.$$

Substituting the expressions for $y_{\text{p. n}}$, $y'_{\text{p. n}}$, $y''_{\text{p. n}}$ in the original equation we get $2Be^{-5x} = 4e^{-5x}$, whence $B = 2$ and so $y_{\text{p. n}} = 2x^2e^{-5x}$. The general solution of the given equation is

$$y(x) = (C_1 + C_2x)\, e^{-5x} + 2x^2e^{-5x}.$$

Example 5. Find the general solution of the equation

$$y'' + 3y' + 2y = x \sin x.$$

The first way. Solution. The characteristic equation $\lambda^2 + 3\lambda + 2 = 0$ has the roots $\lambda_1 = -1$, $\lambda_2 = -2$, therefore

$$y_{\text{g. h}} = C_1e^{-x} + C_2e^{-2x}.$$

Since the number i is not a root of the characteristic equation, the particular solution $y_{\text{p. n}}$ of the nonhomogeneous equation should be sought in the form (see Table 1, case III (1))

$$y_{\text{p. n}} = (A_1x + A_2) \cos x + (B_1x + B_2) \sin x;$$

then

$$y'_{\text{p. n}} = (A_1 + B_2 + B_1x) \cos x + (B_1 - A_2 - A_1x) \sin x,$$
$$y''_{\text{p. n}} = (2B_1 - A_2 - A_1x) \cos x - (2A_1 + B_2 + B_1x) \sin x.$$

Substituting these expressions in the original equation we have

$$(2B_1 - A_2 - A_1x) \cos x - (2A_1 + B_2 + B_1x) \sin x$$
$$+ 3(A_1 + B_2 + B_1x) \cos x + 3(B_1 - A_2 - A_1x) \sin x$$
$$+ 2(A_1x + A_2) \cos x + 2(B_1x + B_2) \sin x = x \sin x$$

or

$$[(A_1 + 3B_1)x + 3A_1 + A_2 + 2B_1 + 3B_2] \cos x$$
$$+ [(-3A_1 + B_1)x - 2A_1 - 3A_2 + 3B_1 + B_2] \sin x$$
$$= x \sin x.$$

Hence we obtain a system of linear equations in A_1, A_2, B_1, B_2

$$\begin{cases} A_1 + 3B_1 = 0, \\ 3A_1 + A_2 + 2B_1 + 3B_2 = 0, \\ -3A_1 + B_1 = 1, \\ -2A_1 - 3A_2 + 3B_1 + B_2 = 0. \end{cases}$$

Solving this system we find $A_1 = -3/10$, $A_2 = 17/50$, $B_1 = 1/10$, $B_2 = 3/25$ and the particular solution $y_{\text{p.n}}$ will be written as

$$y_{\text{p.n}} = \left(-\frac{3}{10}x + \frac{17}{50}\right)\cos x + \left(\frac{1}{10}x + \frac{3}{25}\right)\sin x.$$

The general solution of the given equation is

$$y(x) = C_1 e^{-x} + C_2 e^{-2x} + \left(-\frac{3}{10}x + \frac{17}{50}\right)\cos x + \left(\frac{1}{10}x + \frac{3}{25}\right)\sin x. \blacklozenge$$

Where the right-hand side $f(x)$ contains the trigonometric functions $\sin \beta x$ and $\cos \beta x$ it is found convenient to use a transition to exponential functions. We shall show the essence of this method by considering an example. Solve the differential equation

$$y'' + y = x \cos x.$$

Here $\lambda^2 + 1 = 0$, $\lambda_1 = -i$, $\lambda_2 = i$ and the general solution of the homogeneous equation is of the form

$$y_{\text{g.h}} = C_1 \cos x + C_2 \sin x.$$

The particular solution of the nonhomogeneous equation $y_{\text{p.n}}$ must be sought in the form

$$y_{\text{p.n}} = x\,[(A_1 x + A_2)\cos x + (B_1 x + B_2)\sin x].$$

We proceed as follows. Consider the equation

$$z'' + z = xe^{ix}. \tag{12}$$

It can easily be seen that the right-hand side of the original equation is the real part of the right-hand side of equation (12):

$$x \cos x = \operatorname{Re}(xe^{ix}).$$

Theorem. *If a differential equation with real coefficients $L[y] = f_1(x) + if_2(x)$ has a solution $y = u(x) + iv(x)$, then $u(x)$ is the solution of the equation $L[y] = f_1(x)$ and $v(x)$ is that of the equation $L[y] = f_2(x)$.*

We find $z_{\text{p.n}}$ of equation (12):

$$z_{\text{p.n}} = (Ax + B)\,xe^{ix} = (Ax^2 + Bx)\,e^{ix},$$

$$z''_{\text{p.n}} = 2Ae^{ix} + 2\,(2Ax + B)\,ie^{ix} - (Ax^2 + Bx)\,e^{ix}.$$

Substituting into equation (12) and cancelling e^{ix} from both sides we have

$$2A + 4Axi + 2Bi = x,$$

whence $4Ai = 1$, $A = -i/4$, $A + Bi = 0$, $B = -A/i = 1/4$; so that

$$z_{\text{p.n}} = \left(-\frac{i}{4}x^2 + \frac{1}{4}x\right)e^{ix} = \left(-\frac{i}{4}x^2 + \frac{1}{4}x\right)$$
$$\times(\cos x + i\sin x) = \frac{x\cos x + x^2\sin x}{4} + i\,\frac{x\sin x - x^2\cos x}{4}.$$

Hence by virtue of the theorem

$$y_{\text{p.n}} = \operatorname{Re} z_{\text{p.n}} = \frac{x\cos x + x^2\sin x}{4}.$$

This method now and then considerably simplifies and reduces computations involved in the determination of particular solutions.

The second way of solving Example 5. We shall solve the example by means of transition to exponential functions. Consider the equations

$$z'' + 3z' + 2z = xe^{ix}. \tag{13}$$

It can easily be seen that the right-hand side of the original equation is equal to the imaginary part of xe^{ix}:

$$x\sin x = \operatorname{Im}(xe^{ix}).$$

We seek $z_{\text{p. n}}$ of equation (13) in the form

$$z_{\text{p. n}} = (Ax + B)e^{ix},$$

then

$$z'_{\text{p. n}} = Ae^{ix} + i(Ax + B)e^{ix}, \quad z''_{\text{p. n}} = 2iAe^{ix} - (Ax + B)e^{ix}.$$

Substituting these expressions in (13) and cancelling e^{ix} we get

$$2Ai - Ax - B + 3A + 3Aix + 3Bi + 2Ax + 2B = x$$

whence

$$\begin{cases} A + 3Ai = 1, \\ 2Ai + B + 3A + 3Bi = 0 \end{cases}$$

so that

$$A=\frac{1}{1+3i}=\frac{1}{10}-\frac{3i}{10}, \qquad B=-\frac{A\,(3+2i)}{1+3i}=\frac{6}{50}+\frac{17}{50}\,i.$$

Thus

$$z_{\text{p.n}}=\left(\frac{1-3i}{10}\,x+\frac{6+17i}{50}\right)e^{ix}=\left(\frac{5x+6}{50}+\frac{17-15x}{50}\,i\right)$$

$$\times(\cos x+i\sin x)=\frac{(5x+6)\cos x+(15x-17)\sin x}{50}$$

$$+i\,\frac{(5x+6)\sin x+(17-15x)\cos x}{50};$$

hence

$$y_{\text{p.n}}=\operatorname{Im} z_{\text{p.n}}=\frac{5x+6}{50}\sin x+\frac{17-15x}{50}\cos x$$

which coincides with $y_{\text{p. n}}$ found earlier.

Example 6. Find the general solution of the equation $y''+4y=\sin 2x$.

Solution. Consider the equation $z''+4z=e^{i2x}$.

We have

$$\sin 2x=\operatorname{Im} e^{2ix}, \text{ therefore } y_{\text{p. n}}=\operatorname{Im} z_{\text{p. n}}.$$

The characteristic equation $\lambda^2+4=0$ has simple roots $\lambda_{1,\,2}=\pm 2i$. Consequently, we seek a particular solution in the form (see Table 1, case III (2)):

$$z_{\text{p. n}}=Ax\,e^{2ix},$$

then

$$z''_{\text{p. n}}=-4Ax\,e^{2ix}+4Ai\,e^{2ix}.$$

Substituting the expressions for $z_{\text{p. n}}$ and $z''_{\text{p. n}}$ in the equation and cancelling e^{2ix} we get $4Ai=1$, whence $A=-i/4$ and so

$$z_{\text{p.n}}=-\frac{1}{4}\,ix\,e^{2ix}=\frac{1}{4}\,x\sin 2x-i\,\frac{1}{4}\,x\cos 2x.$$

The particular solution of the given nonhomogeneous equation is

$$y_{\text{p.n}}=\operatorname{Im} z_{\text{p.n}}=-\frac{1}{4}\,x\cos 2x.$$

Example 7. Find the general solution of the equation $y''-6y'+9y=25e^{x}\sin x$.

Solution. The characteristic equation $\lambda^2 - 6\lambda + 9 = 0$ has the roots $\lambda_1 = \lambda_2 = 3$; the general solution $y_{\text{g. h}}$ of the homogeneous equation is

$$y_{\text{g. h}} = (C_1 + C_2 x)\, e^{3x}.$$

The numbers $1 \pm i$ are not roots of the characteristic equation, therefore the particular solution $y_{\text{p. n}}$ should be sought in the form (see Table 1, case IV (1))

$$y_{\text{p. n}} = e^x (a \cos x + b \sin x).$$

Substituting the expression for $y_{\text{p. n}}$ in the equation and cancelling e^x from both sides of the equation we get

$$(3a - 4b) \cos x + (4a + 3b) \sin x = 25 \sin x.$$

Hence we have the system

$$3a - 4b = 0, \quad 4a + 3b = 25$$

whose solution is $a = 4$, $b = 3$ and, consequently,

$$y_{\text{p. n}} = e^x (4 \cos x + 3 \sin x).$$

The general solution of the given equation is

$$y(x) = (C_1 + C_2 x)\, e^{3x} + e^x (4 \cos x + 3 \sin x).$$

Example 8. Find the general solution of the equation $y'' + 2y' + 5y = e^{-x} \cos 2x$.

Solution. The characteristic equation $\lambda^2 + 2\lambda + 5 = 0$ has the roots $\lambda_{1,2} = -1 \pm 2i$, so that

$$y_{\text{g. h}} = (C_1 \cos 2x + C_2 \sin 2x)\, e^{-x}.$$

Since the number $\alpha + i\beta = -1 + 2i$ is a simple root of the characteristic equation, $y_{\text{p. n}}$ must be sought in the form (see Table 1, case IV (2))

$$y_{\text{p. n}} = x (A \cos 2x + B \sin 2x)\, e^{-x}$$

then

$$y'_{\text{p. n}} = e^{-x} [(A - Ax + 2Bx) \cos 2x + (B - Bx - 2Ax) \sin 2x],$$

$$y''_{\text{p. n}} = e^{-x} [(-2A - 3Ax + 4B - 4Bx) \cos 2x + (-2B - 3Bx - 4A + 4Ax) \sin 2x].$$

Substituting the expressions for $y_{\text{p. n}}$ and its derivatives in the original equation and cancelling e^{-x} we have

$$-4A \sin 2x + 4B \cos 2x = \cos 2x,$$

whence $A = 0$, $B = 1/4$ and so

$y_{\text{p. n}} = 1/4x\, e^{-x} \sin 2x.$

The general solution of the given equation is

$y(x) = (C_1 \cos 2x + C_2 \sin 2x)\, e^{-x} + 1/4x\, e^{-x} \sin 2x.$

Determine the form the particular solution of a nonhomogeneous linear differential equation has if the roots of its characteristic equation and its right-hand side $f(x)$ are known:

454. $\lambda_1 = 1,\ \lambda_2 = 2;\ f(x) = ax^2 + bx + c.$

455. $\lambda_1 = 0,\ \lambda_2 = 1;\ f(x) = ax^2 + bx + c.$

456. $\lambda_1 = 0,\ \lambda_2 = 0;\ f(x) = ax^2 + bx + c.$

457. $\lambda_1 = 1,\ \lambda_2 = 2;\ f(x) = e^{-x}(ax + b).$

458. $\lambda_1 = -1,\ \lambda_2 = 1;\ f(x) = e^{-x}(ax + b).$

459. $\lambda_1 = -1,\ \lambda_2 = -1;\ f(x) = e^{-x}(ax + b).$

460. $\lambda_1 = 0,\ \lambda_2 = 1;\ f(x) = \sin x + \cos x.$

461. $\lambda_1 = -i,\ \lambda_2 = i;\ f(x) = \sin x + \cos x.$

462. $\lambda_1 = -2i,\ \lambda_2 = 2i;\ f(x) = A \sin 2x + B \cos 2x.$

463. $\lambda_1 = -ki,\ \lambda_2 = ki;\ f(x) = A \sin kx + B \cos kx.$

464. $\lambda_1 = 1,\ \lambda_2 = 1;\ f(x) = e^{-x}(A \sin x + B \cos x).$

465. $\lambda_1 = -1 - i,\ \lambda_2 = -1 + i;\ f(x) = e^{-x}(A \sin x + B \cos x).$

466. $\lambda_1 = \lambda_2 = \lambda_3 = 1;\ f(x) = ax^2 + bx + c.$

467. $\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 2;\ f(x) = ax^2 + bx + c.$

468. $\lambda_1 = \lambda_2 = 0,\ \lambda_3 = 1;\ f(x) = ax^2 + bx + c.$

469. $\lambda_1 = \lambda_2 = \lambda_3 = 0;\ f(x) = ax^2 + bx + c.$

470. $\lambda_1 = i,\ \lambda_2 = -i,\ \lambda_3 = 1;\ f(x) = \sin x + \cos x.$

471.
$$\left.\begin{array}{l}
\text{(a) } \lambda_1 = 0,\ \lambda_2 = 1, \\
\text{(b) } \lambda_1 = k,\ \lambda_2 = 1, \\
\text{(c) } \lambda_1 = \lambda_2 = k, \\
\text{(d) } \lambda_1 = \lambda_2 = 0,\ \lambda_3 = 1, \\
\text{(e) } \lambda_1 = \lambda_2 = k,\ \lambda_3 = 1, \\
\text{(f) } \lambda_1 = \lambda_2 = \lambda_3 = k
\end{array}\right\}
\begin{array}{l} f(x) = (ax^2 + bx + c)\, e^{kx}, \\ k \neq 0,\ k \neq 1. \end{array}$$

472. (a) $\lambda_1 = \lambda_2 = 1,\ \lambda_3 = 2,$
(b) $\lambda_1 = -i,\ \lambda_2 = i,\ \lambda_3 = 0$ $\Big\}$ $f(x) = a\sin x + b\cos x.$

473. (a) $\lambda_1 = 3 - 2i,\ \lambda_2 = 3 + 2i,$
$\lambda_3 = \lambda_4 = 0,$
(b) $\lambda_1 = \lambda_2 = 3 - 2i,$
$\lambda_3 = \lambda_4 = 3 + 2i$ $\Bigg\}$ $f(x) = e^{3x}(\sin 2x + \cos 2x).$

Determine the form of the particular solution for the following nonhomogeneous linear differential equations:

474. $y'' + 3y' = 3.$

475. $y'' - 7y' = (x - 1)^2.$

476. $y'' + 3y' = e^x.$

477. $y'' + 7y' = e^{-7x}.$

478. $y'' - 8y' + 16y = (1 - x)\,e^{4x}.$

479. $y'' - 10y' + 25y = e^{5x}.$

480. $4y'' - 3y' = x\,e^{\frac{3}{4}x}.$

481. $y'' - 4y' = x\,e^{4x}.$

482. $y'' + 25y = \cos 5x.$

483. $y'' + y = \sin x - \cos x.$

484. $y'' + 16y = \sin(4x + \alpha).$

485. $y'' + 4y' + 8y = e^{2x}(\sin 2x + \cos 2x).$

486. $y'' - 4y' + 8y = e^{2x}(\sin 2x - \cos 2x).$

487. $y'' + 6y' + 13y = e^{-3x}\cos 2x.$

488. $y'' + k^2y = k\sin(kx + \alpha).$

489. $y'' + k^2y = k.$

490. $y''' + y = x.$

491. $y''' + 6y'' + 11y' + 6y = 1.$

492. $y''' + y' = 2.$

493. $y''' + y'' = 3.$

494. $y^{\mathrm{IV}} - y = 1.$

495. $y^{\mathrm{IV}} - y' = 2.$

496. $y^{\mathrm{IV}} - y'' = 3.$

497. $y^{\mathrm{IV}} - y''' = 4$.

498. $y^{\mathrm{IV}} + 4y''' + 4y'' = 1$.

499. $y^{\mathrm{IV}} + 2y''' + y'' = e^{4x}$.

500. $y^{\mathrm{IV}} + 2y''' + y'' = e^{-x}$.

501. $y^{\mathrm{IV}} + 2y''' + y'' = xe^{-x}$.

502. $y^{\mathrm{IV}} + 4y'' + 4y = \sin 2x$.

503. $y^{\mathrm{IV}} + 4y'' + 4y = \cos x$.

504. $y^{\mathrm{IV}} + 4y'' + 4y = x \sin 2x$.

505. $y^{\mathrm{IV}} + 2n^2y'' + n^4y = a \sin (nx + \alpha)$.

506. $y^{\mathrm{IV}} - 2n^2y'' + n^4y = \cos (nx + \alpha)$.

507. $y^{\mathrm{IV}} + 4y''' + 6y'' + 4y' + y = \sin x$.

508. $y^{\mathrm{IV}} - 4y''' + 6y'' - 4y' + y = e^x$.

509. $y^{\mathrm{IV}} - 4y''' + 6y'' - 4y' + y = xe^x$.

Solve the following nonhomogeneous linear equations:

510. $y'' + 2y' + y = -2$.

511. $y'' + 2y' + 2 = 0$.

512. $y'' + 9y - 9 = 0$.

513. $y''' + y'' = 1$.

514. $5y''' - 7y'' - 3 = 0$.

515. $y^{\mathrm{IV}} - 6y''' + 6 = 0$.

516. $3y^{\mathrm{IV}} + y''' = 2$.

517. $y^{\mathrm{IV}} - 2y''' + 2y'' - 2y' + y = 1$.

518. $y'' - 4y' + 4y = x^2$.

519. $y'' + 8y' = 8x$.

520. $y'' - 2ky' + k^2y = e^x$, $(k \neq 1)$.

521. $y'' + 4y' + 4y = 8e^{-2x}$.

522. $y'' + 4y' + 3y = 9e^{-3x}$.

523. $7y'' - y' = 14x$.

524. $y'' + 3y' = 3xe^{-3x}$.

525. $y'' + 5y' + 6y = 10 (1 - x) e^{-2x}$.

526. $y'' + 2y' + 2y = 1 + x$.

527. $y'' + y' + y = (x + x^2)\,e^x$.

528. $y'' + 4y' - 2y = 8\sin 2x$.

529. $y'' + y = 4x\cos x$.

530. $y'' - 2my' + m^2y = \sin nx$.

531. $y'' + 2y' + 5y = e^{-x}\sin 2x$.

532. $y'' + a^2y = 2\cos mx + 3\sin mx$ $(m \neq a)$.

533. $y'' - y' = e^x \sin x$.

534. $y'' + 2y' = 4e^x(\sin x + \cos x)$.

535. $y'' + 4y' + 5y = 10e^{-2x}\cos x$.

536. $4y'' + 8y' = x\sin x$.

537. $y'' - 3y' + 2y = xe^x$.

538. $y'' + y' - 2y = x^2e^{4x}$.

539. $y'' - 3y' + 2y = (x^2 + x)\,e^{3x}$.

540. $y''' - y'' + y' - y = x^2 + x$.

541. $y^{\mathrm{IV}} - 2y''' + 2y'' - 2y' + y = e^x$.

542. $y'' - 2y' + y = x^3$.

543. $y^{\mathrm{IV}} + y'' = x^2 + x$.

544. $y'' + y = x^2\sin x$.

545. $y'' + 2y' + y = x^2e^{-x}\cos x$.

546. $y''' - y = \sin x$.

547. $y^{\mathrm{IV}} - 2y'' + y = \cos x$.

548. $y''' - 3y'' + 3y' - y = e^x\cos 2x$.

549. $y'' - 4y' + 5y = e^{2x}(\sin x + 2\cos x)$.

B. *The superposition principle.* It is convenient to use the following theorem in finding particular solutions of nonhomogeneous linear equations.

Theorem (the superposition principle). *If* $y_k(x)$ *is a solution of the equation*

$$a_0(x)\,y^{(n)} + a_1(x)\,y^{(n-1)} + \dots + a_n(x)\,y = f_k(x),\ k = 1, 2, \dots, m$$

then the function $y(x) = \sum_{k=1}^{m} y_k(x)$ *is a solution of the equation*

$$a_0(x)\, y^{(n)} + a_1(x)\, y^{(n-1)} + \ldots + a_n(x)\, y = \sum_{k=1}^{m} f_k(x).$$

Example 9. Solve the equation

$$y'' - 6y' + 9y = 4e^x - 16e^{3x}. \tag{14}$$

Solution. The characteristic equation $\lambda^2 - 6\lambda + 9 = 0$ has the roots $\lambda_1 = \lambda_2 = 3$ and therefore the general solution $y_{g.h}$ of the corresponding homogeneous equation is

$$y_{g.h} = C_1 e^{3x} + C_2 x e^{3x}.$$

To determine the particular solution $y_{p.n}$ of equation (14) we shall find particular solutions of the following two equations

$$y'' - 6y' + 9y = 4e^x \tag{15}$$

$$y'' - 6y' + 9y = -16e^{3x}. \tag{16}$$

Equation (15) has a particular solution y_1 of the form $y_1 = Ae^x$ (see Table 1, case II (1)). Substituting the expression for y_1 in equation (15) gives $A = 1$, so that $y_1 = e^x$. The particular solution of equation (16) is sought in the form $y_2 = Bx^2e^{3x}$ (see Table 1, case II (2)). We find that $y_2 = -8x^2e^{3x}$.

By virtue of the principle of superposition of solutions the particular solution $y_{p.n}$ of the given equation is the sum of the particular solutions y_1 and y_2 of equations (15) and (16)

$$y_{p.n} = y_1 + y_2 = e^x - 8x^2e^{3x}.$$

The general solution of equation (14) is

$$y = (C_1 + C_2 x)\, e^{3x} + e^x - 8x^2e^{3x}.$$

Example 10. Solve the equation

$$y''' - 2y'' + 2y' = 4 \cos x \cos 3x + 6 \sin^2 x. \tag{17}$$

Solution. Using the well-known trigonometric identities convert the right-hand side of equation (17) to the "standard" form

$$4 \cos x \cos 3x + 6 \sin^2 x = 2 \cos 4x - \cos 2x + 3.$$

The original equation (17) can now be written as

$$y''' - 2y'' + 2y' = 2\cos 4x - \cos 2x + 3. \tag{18}$$

The general solution of the homogeneous equation

$$y''' - 2y'' + 2y' = 0 \text{ is}$$

$$y_{g.h} = C_1 + (C_2 \cos x + C_3 \sin x)\, e^x.$$

To find the particular solution of equation (18) use the superposition principle. To do this find the particular solutions of the following three equations:

$$y''' - 2y'' + 2y' = 2\cos 4x, \tag{19}$$

$$y''' - 2y'' + 2y' = -\cos 2x, \tag{20}$$

$$y''' - 2y'' + 2y' = 3. \tag{21}$$

Using the trial and error method find particular solutions y_1, y_2, and y_3 of equations (19), (20), and (21) respectively:

$$y_1 = \frac{1}{65}\left(\cos 4x - \frac{7}{4}\sin 4x\right),$$

$$y_2 = \frac{1}{10}\left(\frac{1}{2}\sin 2x - \cos 2x\right), \quad y_3 = \frac{3}{2}x.$$

By virtue of the superposition principle the particular solution of the nonhomogeneous equation (18) is

$$y_{p.n} = \frac{1}{65}\left(\cos 4x - \frac{7}{4}\sin 4x\right) + \frac{1}{10}\left(\frac{\sin 2x}{2} - \cos 2x\right) + \frac{3}{2}x.$$

The general solution of the original equation is

$$y = C_1 + (C_2 \cos x + C_3 \sin x)\, e^x + \frac{1}{65}\left(\cos 4x - \frac{7}{4}\sin 4x\right)$$
$$+ \frac{1}{10}\left(\frac{\sin 2x}{2} - \cos 2x\right) + \frac{3}{2}x$$

Determine the form of the particular solution of a nonhomogeneous linear differential equation if we know the roots of its characteristic equation and its right-hand side $f(x)$.

550. $$\left.\begin{array}{ll} \text{(a)} & \lambda_1 = -1,\ \lambda_2 = 1,\ \lambda_3 = 2, \\ \text{(b)} & \lambda_1 = 0,\ \lambda_2 = 2,\ \lambda_3 = 3, \\ \text{(c)} & \lambda_1 = \lambda_2 = -1,\ \lambda_3 = 1, \\ \text{(d)} & \lambda_1 = \lambda_2 = \lambda_3 = -1, \end{array}\right\} \quad f(x) = ae^{-x} + be^{x}.$$

Using the superposition principle determine the form of the particular solution for the following nonhomogeneous linear differential equations:

551. $y'' - y' - 2y = e^x + e^{-2x}$.

552. $y'' + 4y' = x + e^{-4x}$.

553. $y'' - y = x + \sin x$.

554. $y'' - 2y' + 2y = (1 + \sin x) e^x$.

555. $y''' - y'' = 1 + e^x$.

556. $y''' + 4y' = e^{2x} + \sin 2x$.

557. $y'' + 4y = \sin x \sin 2x$.

558. $y'' - 4y' = 2 \cos^2 4x$.

Solve the following nonhomogeneous linear equations using the superposition principle to find their particular solutions:

559. $y'' - y' - 2y = 4x - 2e^x$.

560. $y'' - 3y' = 18x - 10 \cos x$.

561. $y'' - 2y' + y = 2 + e^x \sin x$.

562. $y'' + 2y' + 2y = (5x + 4) e^x + e^{-x}$.

563. $y'' + 2y' + 5y = 4e^{-x} + 17 \sin 2x$.

564. $2y'' - 3y' - 2y = 5e^x \cosh x$.

565. $y'' + 4y = x \sin^2 x$.

566. $y^{IV} + 2y''' + 2y'' + 2y' + y = xe^x + \frac{1}{2} \cos x$.

567. $y'' + y' = \cos^2 x + e^x + x^2$.

568. $y^{V} + 4y''' = e^x + 3 \sin 2x + 1$.

569. $y'' - 2y' + 5y = 10 \sin x + 17 \sin 2x$.

570. $y'' + y' = x^2 - e^{-x} + e^x$.

571. $y'' - 2y' - 3y = 2x + e^{-x} - 2e^{3x}$.

572. $y'' + 4y = e^x + 4 \sin 2x + 2 \cos^2 x - 1$.

573. $y'' + 3y' + 2y = 6xe^{-x} (1 - e^{-x})$.

574. $y'' + y = \cos^2 2x + \sin^2 \frac{x}{2}$.

575. $y'' - 4y' + 5y = 1 + 8 \cos x + e^{2x}$.

576. $y'' - 2y' + 2y = e^x \sin^2 \frac{x}{2}$.

577. $y'' - 3y' = 1 + e^x + \cos x + \sin x$.

578. $y'' - 2y' + 5y = e^x (1 - 2\sin^2 x) + 10x + 1$.

579. $y'' - 4y' + 4y = 4x + \sin x + \sin 2x$.

580. $y'' + 2y' + y = 1 + 2\cos x + \cos 2x - \sin 2x$.

581. $y'' + y' + y + 1 = \sin x + x + x^2$.

582. $y'' + 6y' + 9y = 18e^{-3x} + 8\sin x + 6\cos x$.

583. $y'' + 2y' + 1 = 3\sin 2x + \cos x$.

584. $y''' - 2y'' + y' = 2x + e^x$.

585. $y'' + y = 2\sin x \sin 2x$.

586. $y''' - y'' - 2y' = 4x + 3\sin x + \cos x$.

587. $y''' - 4y' = x e^{2x} + \sin x + x^2$.

588. $y^{\mathrm{V}} - y^{\mathrm{IV}} = x e^x - 1$.

589. $y^{\mathrm{V}} + y''' = x + 2e^{-x}$.

C. *The initial value problem.* It is known that the initial value problem for the nonhomogeneous linear equation

$$y^{(n)} + p_1 y^{(n-1)} + \ldots + p_n y = f(x)$$

is as follows: find the solution of this equation satisfying the initial conditions (the Cauchy data)

$$y(x_0) = y_0,\ y'(x_0) = y_0',\ \ldots,\ y^{(n-1)}(x_0) = y_0^{(n-1)}$$

Example 11. Find the particular solution of the equation

$$y'' - y = 4e^x \tag{22}$$

satisfying the initial conditions

$$y(0) = 0, \quad y'(0) = 1. \tag{23}$$

Solution. We find the general solution of equation (22)

$$y = C_1 e^x + C_2 e^{-x} + 2xe^x. \tag{24}$$

To solve the set initial value problem (22), (23) (the Cauchy problem) it is required that the values of the constants C_1 and C_2 should be determined so that solution (24) satisfies the initial conditions (23). Using the condition $y(0) = 0$ we get $C_1 + C_2 = 0$. Differentiating (24) we

find that

$$y' = C_1 e^x - C_2 e^{-x} + 2e^x + 2x e^x,$$

whence, by virtue of the condition $y'(0) = 1$, we have $C_1 - C_2 = -1$. To find out C_1 and C_2 we have obtained the system

$$\begin{cases} C_1 + C_2 = 0, \\ C_1 - C_2 = -1 \end{cases}$$

solving which gives $C_1 = -1/2$, $C_2 = 1/2$. Substituting the obtained values of the arbitrary constants in the general solution (24) we come to the solution of the initial value problem (22), (23):

$$y = -\frac{1}{2} e^x + \frac{1}{2} e^{-x} + 2xe^x \text{ or}$$

$$y = 2xe^x - \sinh x.$$

Example 12. Find the particular solution of the equation

$$y'' + 4y' + 5y = 8 \cos x, \tag{25}$$

bounded when $x \to -\infty$.

Solution. The general solution of the given equation is

$$y = e^{-2x} (C_1 \cos x + C_2 \sin x) + 2 (\cos x + \sin x) \tag{26}$$

The quantity $e^{-2x} \to +\infty$ as $x \to -\infty$, and for all C_1 and C_2 not simultaneously zero the first right-hand term of (26) is a function unbounded when $x \to -\infty$ and the second term is a function bounded for all values of x. Consequently, only for $C_1 = C_2 = 0$ we have a solution of equation (25), bounded when $x \to -\infty$, namely

$$y = 2 (\cos x + \sin x) \tag{27}$$

and what is more, solution (27) of equation (25) is bounded for all x:

$$| y | = | 2 (\cos x + \sin x) | \leqslant 2 (| \cos x | + | \sin x |) < 4$$

for all $x \in (-\infty, +\infty)$.

Example 13. Find the particular solution of the equation

$$y'' - 3y' + 2y = 4 + 2e^{-x} \cos x \qquad (28)$$

satisfying the condition $y \to 2$ as $x \to +\infty$.

Solution. The general solution of the given equation is

$$y = C_1 e^x + C_2 e^{2x} + 2 + e^{-x} (\sin x - \cos x). \qquad (29)$$

For any values of the constants C_1 and C_2 not simultaneously zero solution (29) is an unbounded function when $x \to +\infty$. For $C_1 = C_2 = 0$ the solution of equation (28) is the function $y = 2 + e^{-x} (\sin x - \cos x)$ for which the condition $\lim\limits_{x \to +\infty} y = 2$ obviously holds. Thus

$$y = 2 + (\sin x - \cos x)\, e^{-x}$$

is the desired particular solution.

In the problems below find the particular solutions of the equations satisfying the given initial conditions:

590. $y'' + y = 2(1 - x)$; $y(0) = 2$, $y'(0) = -2$.

591. $y'' - 6y' + 9y = 9x^2 - 12x + 2$; $y(0) = 1$, $y'(0) = 3$.

592. $y'' + 9y = 36e^{3x}$; $y(0) = 2$, $y'(0) = 6$.

593. $y'' - 4y' + 4y = 2e^{2x}$; $y(0) = y'(0) = 0$.

594. $y'' - 5y' + 6y = (12x - 7)\, e^{-x}$; $y(0) = y'(0) = 0$.

595. $y'' + y' = e^{-x}$; $y(0) = 1$, $y'(0) = -1$.

596. $y'' + 6y' + 9y = 10 \sin x$; $y(0) = y'(0) = 0$.

597. $y'' + y = 2 \cos x$; $y(0) = 1$, $y'(0) = 0$.

598. $y'' + 4y = \sin x$; $y(0) = y'(0) = 1$.

599. $y'' + y = 4x \cos x$; $y(0) = 0$, $y'(0) = 1$.

600. $y'' - 4y' + 5y = 2x^2 e^x$; $y(0) = 2$, $y'(0) = 3$.

601. $y'' - 6y' + 9y = 16e^{-x} + 9x - 6$; $y(0) = y'(0) = 1$.

602. $y'' - y' = -5e^{-x} (\sin x + \cos x)$; $y(0) = -4$, $y'(0) = 5$.

603. $y'' - 2y' + 2y = 4e^x \cos x$; $y(\pi) = \pi e^{\pi}$, $y'(\pi) = e^{\pi}$.

604. $y''' - y' = -2x$; $y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$.

605. $y^{IV} - y = 8e^x$; $y(0) = -1$, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 0$.

606. $y''' - y = 2x$; $y(0) = y'(0) = 0$, $y''(0) = 2$.

607. $y^{IV} - y = 8e^x$; $y(0) = 0$, $y'(0) = 2$, $y''(0) = 4$, $y'''(0) = 6$.

In the problems below find the particular solutions of the equations satisfying the given conditions at infinity:

608. $y'' - 4y' + 5y = \sin x$, y bounded when $x \to +\infty$

609. $y'' + 2y' + 5y = 4\cos 2x + \sin 2x$, y bounded when $x \to -\infty$.

610. $y'' - y = 1$, y bounded when $x \to \infty$.

611. $y'' - y = -2\cos x$, y bounded when $x \to \infty$.

612. $y'' - 2y' + y = 4e^{-x}$, $y \to 0$ as $x \to +\infty$.

613. $y'' + 4y' + 3y = 8e^x + 9$, $y \to 3$ as $x \to -\infty$.

614. $y'' - y' - 5y = 1$, $y \to -\frac{1}{5}$ as $x \to \infty$.

615. $y'' + 4y' + 4y = 2e^x(\sin x + 7\cos x)$, $y \to 0$ as $x \to -\infty$.

616. $y'' - 5y' + 6y = 2e^{-2x}(9\sin 2x + 4\cos 2x)$, $y \to 0$ as $x \to +\infty$.

617. $y'' - 4y' + 4y = (9x^2 + 5x - 12)e^{-x}$, $y \to 0$ as $x \to +\infty$.

15.4. The Euler equations. Linear equations of the form

$$a_0x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + \ldots + a_{n-1}xy' + a_ny = 0, \quad (30)$$

where all a_i are constant, are given the name of *Euler equations*. These, by the substitution of the independent variable $x = e^t$, can be transformed into homogeneous linear equations with constant coefficients:

$$b_0y_t^{(n)} + b_1y_t^{(n-1)} + \ldots + b_{n-1}y_t' + b_ny(t) = 0. \quad (31)$$

Remark 1. Equations of the form

$$a_0(ax+b)^n y^{(n)} + a_1(ax+b)^{n-1}y^{(n-1)} + \ldots$$
$$\ldots + a_{n-1}(ax+b)y' + a_ny = 0$$

are also called Euler equations and can be reduced to homogeneous linear equations with constant coefficients by the substitution of the variables $ax + b = e^t$.

Remark 2. Particular solutions of equation (30) can at once be sought in the form $y = x^k$, giving for k a solution which coincides with the characteristic equation for equation (31).

Example 1. Find the general solution of the Euler equation $x^2y'' + 2xy' - 6y = 0$.

The first way. We use the substitution $x = e^t$ in the equation, then

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = e^{-t}\frac{dy}{dt}.$$

$$y'' = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)e^{-t}}{e^t} = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

and the equation takes the form

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 0.$$

The roots of the characteristic equation are $\lambda_1 = -3$, $\lambda_2 = 2$ and the general solution of the last equation is $y = C_1e^{-3t} + C_2e^{2t}$. But since $x = e^t$, $y = C_1x^{-3} + C_2x^2$ or

$$y = \frac{C_1}{x^3} + C_2x^2.$$

The second way. We shall seek the solution of the given equation in the form $y = x^k$, where k is an unknown number. We get $y' = kx^{k-1}$, $y'' = k(k-1)x^{k-2}$. Substituting into the equation we have

$$x^2k(k-1)x^{k-2} + 2xkx^{k-1} - 6x^k = 0$$

or

$$x^k[k(k-1) + 2k - 6] = 0.$$

But since $x^k \not\equiv 0$, we find that $k(k-1) + 2k - 6 = 0$ or $k^2 + k - 6 = 0$. The roots of the equation are $k_1 = -3$, $k_2 = 2$. The corresponding fundamental system of solutions $y_1 = x^{-3}$, $y_2 = x^2$ and the general solution is as before

$$y = C_1x^{-3} + C_2x^2. \blacklozenge$$

Nonhomogeneous Euler equations of the form

$$\sum_{k=0}^{n} a_k x^k y^{(k)} = x^{\alpha} P_m (\ln x),$$

where $P_m(u)$ is a polynomial of degree m, can also be solved by the trial and error method by analogy with the solution of the nonhomogeneous linear differential equation with constant coefficients and the right-hand side of the form $e^{\alpha x} P_m(x)$.

Example 2. Solve the Euler equation $x^2y'' - xy' + 2y = x \ln x$.

Solution. The characteristic equation $k(k-1) - k + 2 = 0$ or $k^2 - 2k + 2 = 0$ has the roots $k_1 = 1 - i$, $k_2 = 1 + i$. Therefore the general solution of the corresponding homogeneous equation is

$$y_{\text{g.h}} = x (C_1 \cos \ln x + C_2 \sin \ln x).$$

The particular solution is sought in the form $y_p = x(A \ln x + B)$; we have

$$y_p' = A \ln x + B + a, \quad y_p'' = A/x.$$

Substituting into the given equation we get

$$Ax - x(A \ln x + A + B) + 2x(A \ln x + B) = x \ln x$$

or

$$Ax \ln x + Bx = x \ln x$$

whence $A = 1$, $B = 0$; so $y_p = x \ln x$.

The general solution is

$$y = x (C_1 \cos \ln x + C_2 \sin \ln x) + x \ln x.$$

Integrate the following homogeneous Euler equations:

618. $x^2y'' + xy' - y = 0$.

619. $x^2y'' + 3xy' + y = 0$.

620. $x^2y'' + 2xy' + 6y = 0$.

621. $xy'' + y' = 0$.

622. $(x+2)^2 y'' + 3(x+2) y' - 3y = 0$.

623. $(2x+1)^2 y'' - 2(2x+1) y' + 4y = 0$.

624. $x^2y''' - 3xy'' + 3y' = 0$.

625. $x^2y''' = 2y'$.

626. $(x+1)^4 y''' - 12y' = 0.$

627. $(2x+1)^2 y''' + 2(2x+1) y'' + y' = 0.$

Solve the following nonhomogeneous Euler equations:

628. $x^2y'' + xy' + y = x(6 - \ln x).$

629. $x^2y'' - 2y = \sin \ln x.$

630. $x^2y'' - xy' - 3y = -\frac{16 \ln x}{x}.$

631. $x^2y'' - 2xy' + 2y = x^2 - 2x + 2.$

632. $x^2y'' + xy' - y = x^m, \ |m| \neq 1.$

633. $x^2y'' + 4xy' + 2y = 2\ln^2 x + 12x.$

634. $(x+1)^3 y'' + 3(x+1)^2 y' + (x+1) y = 6 \ln (x+1).$

635. $(x-2)^2 y'' - 3(x-2) y' + 4y = x.$

15.5. Linear differential equations with variable coefficients. The Lagrange method. If one knows the particular solution $y_1(x)$ of the equation

$$y^{(n)} + p_1(x) y^{(n-1)} + \ldots + p_n(x) y = 0 \qquad (32)$$

then one can depress its order by one (without loss of linearity of the equation) by first setting $y = y_1z$, where z is a new unknown function, and then making the substitution $z' = u$ [it is possible to use the direct substitution $u = (y/y_1)'$].

If one knows k particular linearly independent solutions of equation (32), one can depress the order of the equation by k units.

The general solution of the equation

$$y^{(n)} + p_1(x) y^{(n-1)} + \ldots + p_n(x) y = f(x) \qquad (33)$$

is the sum of one of its particular solutions and the general solution of the corresponding homogeneous equation (32).

If the fundamental system of the corresponding homogeneous equation (32) is known, then it is possible to find the general solution of the nonhomogeneous equation (33) by the method of variation of parameters (*the Lagrange method*).

The general solution of equation (32) is of the form

$$y = C_1y_1 + C_2y_2 + \ldots + C_ny_n,$$

$C_1, C_2, \ldots, C_n$ being arbitrary constants.

We shall seek the solution of equation (33) in the form

$$y = C_1(x)\,y_1 + C_2(x)\,y_2 + \ldots + C_n(x)\,y_n, \qquad (34)$$

where $C_1(x)$, $C_2(x)$, . . ., $C_n(x)$ are some as yet unknown functions of x. In order to determine them we obtain the following system

$$\begin{cases} y_1C_1' + y_2C_2' + \ldots + y_nC_n' = 0, \\ y_1'C_1' + y_2'C_2' + \ldots + y_n'C_n' = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_1^{(n-1)}C_1' + y_2^{(n-1)}C_2' + \ldots + y^{(n-1)}C_n' = f(x). \end{cases} \qquad (35)$$

Resolving this system for $C\ (x)$, $i = 1, 2, \ldots, n$ we get

$$\frac{dC_i}{dx} = \varphi_i(x), \quad i = 1, 2, \ldots, n,$$

whence

$$C_i(x) = \int \varphi_i(x)\,dx + \widetilde{C}_i, \quad i = 1, 2, \ldots, n,$$

where $\widetilde{C}_i$ are arbitrary constants. Substituting the obtained values of $C_i(x)$ in (34) we find the general solution of equation (33).

In particular, for the second order equation

$$y'' + p_1(x)\,y' + p_2(x)\,y = f(x).$$

System (35) is of the form

$$\begin{cases} y_1C_1' + y_2C_2' = 0, \\ y_1'C_1' + y_2'C_2' = f(x). \end{cases} \qquad (36)$$

Solving (36) for C_1' and C_2' we get

$$C_1' = -\frac{y_2 f(x)}{W[y_1, y_2]}, \quad C_2' = \frac{y_1 f(x)}{W[y_1, y_2]},$$

whence we find

$$C_1(x) = -\int \frac{y_2 f(x)}{W[y_1, y_2]}\,dx + \widetilde{C}_1', \quad C_2(x) = \int \frac{y_1 f(x)}{W[y_1, y_2]}\,dx + \widetilde{C}_2',$$

$\widetilde{C}_1$ and $\widetilde{C}_2$ being integration constants.

Remark. For the equation $a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = f(x)$, where $a_0(x) \not\equiv 1$, $a_0(x) \neq 0$, system (36) will

look thus:

$$\begin{cases} y_1C_1' + y_2C_2' = 0, \\ y_1'C_1' + y_2'C_2' = \dfrac{f(x)}{a_0(x)}. \end{cases}$$

Example 1. Find the general solution of the equation $xy'' + 2y' + xy = 0$, if $y_1 = \dfrac{\sin x}{x}$ is its particular solution.

Solution. We set $y = \dfrac{\sin x}{x} z$, where z is a new unknown function of x; then

$$y' = y_1'z + y_1z', \quad y'' = y_1''z + 2y_1'z' + y_1z''$$

Substituting into the given equation we get

$$(xy_1'' + 2y_1' + xy_1)z + xy_1z'' + 2(xy_1' + y_1)z' = 0.$$

But since $y_1 = \dfrac{\sin x}{x}$ is a particular solution of the given equation, we have $xy_1'' + 2y_1' + xy_1 = 0$; therefore

$$xy_1z'' + 2(xy_1' + y_1)z' = 0. \tag{37}$$

But $y_1' = \dfrac{\cos x}{x} - \dfrac{\sin x}{x^2}$ and hence $xy_1' + y_1 = \cos x$ and equation (37) will take the form

$$z'' \sin x + 2z' \cos x = 0.$$

We rewrite it as

$$\frac{z''}{z'} + 2\frac{\cos x}{\sin x} = 0.$$

Hence we have $(\ln|z'| + 2\ln|\sin x|)' = 0$ giving

$$\ln|z'| + 2\ln|\sin x| = \ln \widetilde{C}_1 \text{ or } z' \sin^2 x = \widetilde{C}_1.$$

Integrating this equation we find that $z = -\widetilde{C}_1 \cot x + C_2$ and, consequently, the general solution of the given equation is

$$y = -\widetilde{C}_1 \frac{\cos x}{x} + C_2 \frac{\sin x}{x}$$

or

$$y = C_1 \frac{\cos x}{x} + C_2 \frac{\sin x}{x} \qquad (C_1 = -\widetilde{C}_1).$$

Example 2. Find the general solution of the equation $y'' + \dfrac{2}{x} y' + y = \dfrac{1}{x}$, $(x \neq 0)$.

Solution. The general solution of the corresponding homogeneous equation is of the form (see Example 1)

$$y_{g.h} = C_1 \frac{\sin x}{x} + C_2 \frac{\cos x}{x}$$

and, consequently, its fundamental system of solutions is

$$y_1 = \frac{\sin x}{x}, \quad y_2 = \frac{\cos x}{x}.$$

We shall seek the general solution of the given equation by the method of variation of arbitrary parameters:

$$y = C_1(x) \frac{\sin x}{x} + C_2(x) \frac{\cos x}{x},$$

where $C_1(x)$, $C_2(x)$ are as yet unknown functions of x subject to determination. In order to find them we set up the following system:

$$\begin{cases} C_1'(x) \dfrac{\sin x}{x} + C_2'(x) \dfrac{\cos x}{x} = 0, \\ C_1'(x) \dfrac{x\cos x - \sin x}{x^2} + C_2'(x) \dfrac{-x \sin x - \cos x}{x^2} = \dfrac{1}{x}. \end{cases}$$

Hence we find: $C_1'(x) = \cos x$, $C_2'(x) = -\sin x$. Integrating we get

$$C_1(x) = \sin x + \widetilde{C}_1, \quad C_2(x) = \cos x + \widetilde{C}_2.$$

Substituting these values of $C_1(x)$ and $C_2(x)$ in the expression for y we find the general solution of the given equation

$$y = \widetilde{C}_1 \frac{\sin x}{x} + \widetilde{C}_2 \frac{\cos x}{x} + \frac{1}{x}.$$

Example 3. Solve the equation $y'' + y = 1/\cos x$.

Solution. The corresponding homogeneous equation is $y'' + y = 0$. Its characteristic equation $\lambda^2 + 1 = 0$ has the imaginary roots $\lambda_1 = -i$, $\lambda_2 = i$ and the general solution of the homogeneous equation is of the form

$$y_{g.h} = C_1 \cos x + C_2 \sin x.$$

The general solution of the original equation is sought in the form

$$y = C_1(x) \cos x + C_2(x) \sin x, \tag{38}$$

where $C_1(x)$ and $C_2(x)$ are unknown functions of x. To find them we set up the following system

$$\begin{cases} \cos x \times C_1'(x) + \sin x \times C_2'(x) = 0, \\ -\sin x \times C_1'(x) + \cos x \times C_2'(x) = \dfrac{1}{\cos x}. \end{cases}$$

We resolve the system for $C_1'(x)$ and $C_2'(x)$:

$$C_1'(x) = -\tan x; \quad C_2'(x) = 1.$$

By integrating we find

$$C_1(x) = \ln|\cos x| + \tilde{C}_1, \quad C_2(x) = x + \tilde{C}_2.$$

Substituting the expressions for $C_1(x)$ and $C_2(x)$ in (38) we obtain the general solution of the given equation

$$y = \tilde{C}_1 \cos x + \tilde{C}_2 \sin x + \cos x \cdot \ln|\cos x| + x \sin x.$$

Here $\cos x \ln|\cos x| + x \sin x$ is the particular solution of the original nonhomogeneous equation.

Example 4. Given the fundamental system of solutions $y_1 = \ln x$, $y_2 = x$ of the corresponding homogeneous equation, find the particular solution of the equation

$$x^2(1-\ln x)y'' + xy' - y = \frac{(1-\ln x)^2}{x}, \tag{39}$$

satisfying the condition $\lim\limits_{x\to+\infty} y = 0$.

Solution. Using the method of variation of parameters we find the general solution of equation (39):

$$y = C_1 \ln x + C_2 x + \frac{1-2\ln x}{4x}. \tag{40}$$

The first two right-hand terms of (40) go into infinity as $x \to +\infty$, for any C_1, C_2 not simultaneously zero the function $C_1 \ln x + C_2 x$ being an infinitely large function when $x \to +\infty$. The third right-hand term of (40) has zero as the limit when $x \to +\infty$, which is easy to establish using the L'Hospital rule. Thus the function $y = (1 - 2\ln x)/4x$ obtained from (40) when $C_1 = 0$ and $C_2 = 0$ is the solution of equation (39) satisfying the condition $\lim\limits_{x\to+\infty} y = 0$.

Integrate the following equations if one particular solution y_1 of the homogeneous equation is known.

636. $(2x + 1)y'' + (4x - 2)y' - 8y = 0$; $y_1 = e^{mx}$.

637. $(x^2 - x) y'' + (2x - 3) y' - 2y = 0$; y_1 is a rational fraction whose denominator contains linear multipliers dividing the coefficient of y''.

638. $(3x + 2x^2) y'' - 6(1 + x) y' + 6y = 6$; y_1 is a polynomial.

639. $x^2 (\ln x - 1) y'' - xy' + y = 0$, $y_1 = x$.

640. $y'' + (\tan x - 2\cot x) y' + 2\cot^2 x \cdot y = 0$; $y_1 = \sin x$.

641. $y'' + \tan x \cdot y' + \cos^2 x \cdot y = 0$; $y_1 = \cos(\sin x)$.

642. $(1 + x^2) y'' + xy' - y + 1 = 0$; $y_1 = x$.

643. $x^2y'' - xy' - 3y = 5x^4$, $y_1 = \frac{1}{x}$.

644. $(x - 1) y'' - xy' + y = (x - 1)^2 e^x$; $y_1 = e^x$.

645. $y'' + y' + e^{-2x}\ y = e^{-3x}$; $y_1 = \cos e^{-x}$.

646. $(x^4 - x^3) y'' + (2x^3 - 2x^2 - x) y' - y = \frac{(x-1)^2}{x}$; $y_1 = \frac{1}{x}$.

647. $y'' - y' + ye^{2x} = xe^{2x} - 1$, $y_1 = \sin e^x$.

648. $x(x - 1) y'' - (2x - 1) y' + 2y = x^2 (2x - 3)$, $y_1 = x^2$.

649. A chain 6 m long slides off a table without friction. If the motion starts at the moment when 1 m of the chain hangs down, what time is it required for the whole of the chain to slide off?

650. Find the equation of the motion of a point, if the acceleration as a function of time is expressed by the formula $a = 1.2\,t$ and if when $t = 0$ the distance is $s = 0$ and when $t = 5$ the distance is $s = 20$.

651. A body of mass m slides along a horizontal plane under a push which has imparted the initial velocity v_0. The body is acted on by a force of friction equal to km. Find the distance which the body is able to travel.

652. A particle of mass $m = 1$ moves in a straight line approaching a centre repulsing it with a force equal to k^2x (x being the distance of the point from the centre). When $t = 0$, $x = a$, $\frac{dx}{dt} = ka$. Find the law of motion.

Integrate the following equations by the method of variation of parameters:

653. $y''+y=\frac{1}{\sin x}$.

654. $y''-y'=\frac{1}{e^x+1}$.

655. $y''+y=\frac{1}{\cos^3 x}$.

656. $y''+y=\frac{1}{\sqrt{\sin^5 x\cos x}}$.

657. $y''-2y'+y=\frac{e^x}{x^2+1}$.

658. $y''+2y'+2y=\frac{1}{e^x\sin x}$.

659. $y''+y=\frac{2}{\sin^3 x}$.

660. $y''-y'=e^{2x}\cos e^x$.

661. $y'''+y''=\frac{x-1}{x^2}$.

662. $xy''-1(1+2x^2)\,y'=4x^3e^{x^2}$.

663. $y''-2y'\cdot\tan x=1$.

664. $x\ln x\cdot y''-y'=\ln^2 x$.

665. $xy''+(2x-1)\,y'=-4x^2$.

666. $y''+y'\tan x=\cos x\cot x$.

Find the solutions of the following differential equations for the given conditions at infinity:

667. $4xy''+2y'+y=1,\quad \lim\limits_{x\to+\infty} y=1;$

$y_1=\sin\sqrt{x},\quad y_2=\cos\sqrt{x}$.

668. $4xy''+2y'+y=\frac{6+x}{x^2},\quad \lim\limits_{x\to+\infty} y=0$.

669. $(1+x^2)\,y''+2xy'=\frac{1}{1+x^2},\quad \lim\limits_{x\to+\infty} y=\frac{\pi^2}{8},$

$y'\,|_{x=0}=0$.

670. $(1-x)\,y''+xy'-y=(x-1)^2\,e^x,\quad \lim\limits_{x\to-\infty} y=0,$

$y\,|_{x=0}=1;\quad y_1=x,\quad y_2=e^x$.

671. $2x^2(2-\ln x)\,y''+x(4-\ln x)\,y'-y=\dfrac{(2-\ln x)^2}{\sqrt{x}}$,

$\lim\limits_{x\to+\infty} y=0,\quad y_1=\ln x,\quad y_2=\sqrt{x}.$

672. $y''+\dfrac{2}{x}y'-y=4e^x,\quad \lim\limits_{x\to-\infty} y=0,\quad y'\,|_{x=-1}=-\dfrac{1}{e}$;

$y_1=\dfrac{e^x}{x},\quad y_2=\dfrac{e^{-x}}{x}.$

673. $x^3(\ln x-1)\,y''-x^2y'+xy=2\ln x$,

$\lim\limits_{x\to+\infty} y=0,\quad y_1=x,\quad y_2=\ln x.$

674. $(x^2-2x)\,y''+(2-x^2)\,y'-2(1-x)\,y=2(x-1)$,

$\lim\limits_{x\to+\infty} y=1,\quad y_1=x^2,\quad y_2=e^x.$

15.6. Forming a differential equation from a given fundamental system of solutions. Consider a system of functions

$$y_1(x),\ y_2(x),\ \ldots,\ y_n(x) \tag{41}$$

having all derivatives to the nth order inclusively, which is linearly independent in the interval $[a, b]$. Then the equation

$$\begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) & y(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) & y'(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_1^{(n)}(x) & y_2^{(n)}(x) & \ldots & y_n^{(n)}(x) & y^{(n)}(x) \end{vmatrix}=0, \tag{42}$$

where $y(x)$ is an unknown function, will be a linear differential equation for which, as one can easily see, the functions $y_1(x), y_2(x), \ldots, y_n(x)$ constitute a fundamental system of solutions. The coefficient of $y^{(n)}(x)$ in (42) is the Wronskian $W[y_1, y_2, \ldots, y_n]$ of system (41). The points at which the determinant vanishes are singularities of the equation constructed, at these points the coefficient of the higher derivative $y^{(n)}(x)$ vanishes.

Example 1. Set up a differential equation for which the functions $y_1(x)=e^x$, $y_2(x)=e^{-x}$ form a fundamental system of solutions.

Solution. Using the formula (42) we get

$$\begin{vmatrix} e^{x} & e^{-x} & y \\ e^{x} & -e^{-x} & y' \\ e^{x} & e^{-x} & y'' \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 1 & 1 & y \\ 1 & -1 & y' \\ 1 & 1 & y'' \end{vmatrix} = 0. \tag{43}$$

Expanding the determinant in the left-hand side of (43) using the third column we have $y'' - y = 0$. This is exactly the desired differential equation.

Example 2. Set up a differential equation for which the fundamental system of solutions is formed by the functions $y_1(x) = e^{x^2}$, $y_2(x) = e^{-x^2}$.

Solution. We set up an equation of the form (42):

$$\begin{vmatrix} e^{x^2} & e^{-x^2} & y \\ 2x\,e^{x^2} & -2x\,e^{-x^2} & y' \\ (2+4x^2)\,e^{x^2} & (4x^2-2)\,e^{-x^2} & y'' \end{vmatrix} = 0 \quad \text{or}$$

$$\begin{vmatrix} 1 & 1 & y \\ 2x & -2x & y' \\ 2+4x^4 & 4x^2-2 & y'' \end{vmatrix} = 0.$$

Expanding the last determinant using the third column we have

$$xy'' - y' - 4x^3y = 0. \tag{44}$$

In this example the Wronskian $W[y_1, y_2] = -4x$ vanishes when $x = 0$. This does not contradict the general theory by virtue of which the Wronskian of the fundamental system of solutions of a homogeneous linear differential equation

$$y^{(n)} + p_1(x)\,y^{(n-1)} + \ldots + p_n(x)\,y = 0$$

with coefficients continuous in the interval $[a, b]$ does not vanish at any point x of the interval $[a, b]$. On writing equation (44) as

$$y'' - \frac{1}{x}\,y' - 4x^2y = 0 \tag{45}$$

we see that the coefficient of y' becomes discontinuous when $x = 0$, so that at the point $x = 0$ the continuity of the coefficients of equation (45) is broken.

Set up the differential equations for which the given systems of functions form the fundamental system of solutions:

675. $y_1(x) = \sinh x$, $y_2(x) = \cosh x$.

676. $y_1(x) = x$, $y_2(x) = e^x$.

677. $y_1(x) = e^x$, $y_2(x) = e^{x^2/2}$.

678. $y_1(x) = 1$, $y_2(x) = x$, $y_3(x) = x^2$.

679. $y_1(x) = x$,, $y_2(x) = \sin x$, $y_3(x) = \cos x$.

15.7. Miscellaneous problems. Let $y_1, y_2, \ldots, y_n$ be the fundamental system of the homogeneous linear equation

$$y^{(n)} + p_1(x)\, y^{(n-1)} + \ldots + p_n(x)\, y = 0.$$

Then the Ostrogradsky-Liouville formula

$$W(x) = W(x_0)\, e^{-\int\limits_{x_0}^{x} p_1(t)\,dt}$$

holds, $W(x) = W[y_1, y_2, \ldots, y_n]$ being the Wronskian and x_0 any value of x in the interval $[a, b]$ in which the coefficients $p_1(x), p_2(x), \ldots, p_n(x)$ of the equation are continuous.

Example 1. Show that the linear differential equation $xy'' - (x + 2)\, y' + y = 0$ has a solution of the form $y_1 = P(x)$, where $P(x)$ is some polynomial. Show that a second solution y_2 of this equation is of the form $y_2 = e^x Q(x)$, where $Q(x)$ is also a polynomial.

Solution. We shall seek the solution $y_1(x)$ in the form of a polynomial, of the first degree for example: $y_1 = Ax + B$. Substituting into the equation we find that $-2A + B = 0$. Let $A = 1$, then $B = 2$; thus the polynomial $y_1 = x + 2$ is the solution of the given equation. We rewrite the given equation as

$$y'' - \frac{x+2}{x}\, y' + \frac{1}{x}\, y = 0.$$

Let $y_2(x)$ be the second particular solution of the given equation linearly independent together with the first. We find the Wronskian of the system of solutions $y_1 = x + 2$, y_2:

$$W[y_1, y_2] = \begin{vmatrix} x+2 & y_2 \\ 1 & y_2' \end{vmatrix} = (x + 2)\, y_2' - y_2,$$

here $x \neq -2$. Using the Ostrogradsky-Liouville formula we have

$$(x+2)\,y_2' - y_2 = W(x_0)\,e^{\int\limits_{x_0}^{x} \frac{t+2}{t}\,dt},$$

where x_0 is any value of x, with $x_0 \neq 0$, $x_0 \neq -2$, or

$$(x+2)\,y_2' - y_2 = Ax^2 e^x;$$

here $A = \frac{W(x_0)\,e^{-x_0}}{x_0^2} = \text{const}$.
We have obtained a first order linear differential equation to find y_2. Dividing both sides of this equation by $(x+2)^2$ brings it into the form

$$\left(\frac{y_2}{x+2}\right)' = A\frac{x^2 e^x}{(x+2)^2}.$$

Integrating we find that

$$\frac{y_2}{x+2} = A\frac{x-2}{x+2}e^x; \quad \text{hence } y_2 = A(x-2)\,e^x.$$

680. Show that the linear differential equation $(x^2-1) \times y'' = 2y$ has a polynomial solution $y_1(x) = P(x)$. Show that a second solution $y_2(x)$ of this equation is of the form

$$y_2(x) = P(x)\ln\frac{x+1}{x-1} + Q(x),$$

whehe $Q(x)$ is also a polynomial.

681. Find the general solution of the second order homogeneous linear differential equation $y'' + p_1(x)\,y' + p_2(x)\,y = 0$ if we know one of its particular solutions $y_1 = y_1(x)$.

682. Let $y_1(x)$, $y_2(x)$ be the fundamental system of solutions of the second order homogeneous linear differential equation

$$p_0(x)\,y'' + p_1(x)\,y' + p_2(x)\,y = 0.$$

Express the coefficients $p_0(x)$, $p_1(x)$, $p_2(x)$ in terms of $y_1(x)$ and $y_2(x)$.

683. Prove that two solutions of the equation $y'' + p_1(x)y' + p_2(x)\,y = 0$ with continuous coefficients having maximum for one and the same value of x are linearly dependent.

684. Prove that the ratio of any two linearly independent solutions of the equation $y'' + p_1(x)\,y' + p_2(x)\,y = 0$

with continuous coefficients cannot have points of local maximum.

685. For what values of p_1 and p_2 does each solution of the equation $y'' + p_1y' + p_2y = 0$ (p_1, p_2 = const) vanish on an infinite set of points x?

686. Let functions $u(x)$ and $v(x)$ be respectively the solutions of the equations $u'' + p(x)u = 0$ and $v'' + q(x)v = 0$ satisfying the condition $u(a) = 0$, $v(a) = 0$ ($p(x)$ and $q(x)$ being continuous in the interval $[a, b]$). Prove that the Wronskian of these solutions is

$$W[u(x), v(x)] = \int_a^x [p(t) - q(t)]\, u(t)\, v(t)\, dt.$$

687. Prove that no two linearly independent solutions $y_1(x)$ and $y_2(x)$ of the homogeneous linear equation $y'' + p(x)y' + q(x)y = 0$ can vanish together at the same point x_0.

688. Prove that if $y_1(x)$ is some particular solution of the homogeneous linear equation $y'' + p(x)y' + q(x)y = 0$, with $y_1(x) \not\equiv 0$, then the equation $y_1(x) = 0$ cannot have multiple roots.

689. Show that the substitution $y = v(x)z(x)$ transforms the homogeneous linear equation $y'' + p(x)y' + q(x)y = 0$ again into a homogeneous linear equation. How is the function $v(x)$ to be selected for the transformed equation to contain no term with the first derivative?

690. Prove that the solution of the equation $\frac{d^2x}{dt^2} + k^2x = f(t)$ satisfying the initial conditions $x(0) = 0$, $x'(0) = 0$ is of the form

$$x(t) = \frac{1}{k}\int_0^t f(u) \sin k(t-u)\, du.$$

691. For what values of p and q do all solutions of the equation $y'' + py' + qy = 0$ tend to zero as $x \to +\infty$?

692. For what values of p and q are all solutions of the equation $y'' + py' + qy = 0$ (p, q = const) periodic functions of x?

693. Let the function $y(x)$ be a solution of the equation $(1 + x^2)y'' - x^3y' = x^2 + 4$ in the interval $[a, b]$, this

solution satisfying the boundary conditions $y(a) = 0$, $y(b) = 0$.

Prove that for all x in the interval (a, b) $y(x) < 0$.

694. Prove that in the case $q(x) < 0$ no solutions of the equation $y'' + p(x) y' + q(x) y = 0$ can have positive maxima.

695. Prove that in the case $q(x) > 0$ for any solution of the equation $y'' + q(x) y = 0$ the ratio $y'(x)/y(x)$ decreases as x increases in the interval where $y(x) \neq 0$.

16. The method of isoclines for differential equations of the second order

The method of isoclines (see Sec. 2) is also applied to solve some equations of the second order. These are the equations that can be reduced to first order equations, for example equations of the form

$$\frac{d^2x}{dt^2} + f\left(\frac{dx}{dt}, x\right) = 0. \tag{1}$$

We introduce a new variable $v = \frac{dx}{dt}$. Then $\frac{d^2x}{dt^2} = v\frac{dv}{dx}$ and equation (1) takes the form

$$\frac{dv}{dx} = -\frac{f(v, x)}{v}. \tag{2}$$

This is an equation of the first order in which x is an independent variable and one can use the method of isoclines to solve it. We shall think of x as the motion of some point of a system and of $\frac{dx}{dt}$ as its velocity.

The plane of the variables x, v is called *a phase plane*. Thus equation (2) defines velocity as a function of motion. By constructing a field of isoclines for equation (2) we can draw an integral curve, given a starting point (x_0, v_0). This graphical representation of velocity v as a function of motion x: $v = v(x)$ is called *a phase pattern*. The curves in the x, v plane depicting this relation are called *phase trajectories*. Instantaneous values of x and v are the coordinates of a point of a phase trajectory. This point is called a *representative point*. The representative point moves along the phase trajectory with time. Note that positive velocity causes an increase in motion with time. Indeed, by virtue

of the substitution, $v = \frac{dx}{dt}$ for $v > 0$ and $\frac{dx}{dt} > 0$, which means that x increases as t increases. Thus the representative point must move from left to right in the upper half of the phase plane, where $v > 0$, and from right to left in the lower

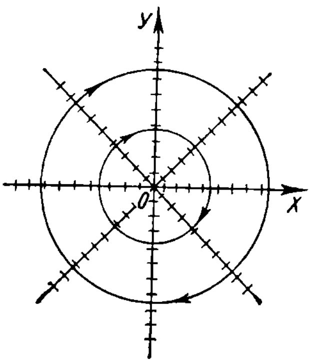

Fig. 26

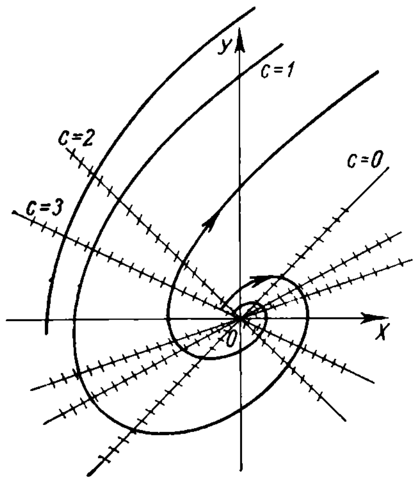

Fig. 27

half of the plane where $v < 0$. Therefore the motion is clockwise in the phase trajectory.

Example 1. Construct trajectories in the phase plane for the equation

$$\frac{d^2x}{dt^2} + x = 0. \tag{3}$$

Solution. We set $\frac{dx}{dt} = v$. Equation (3) assumes the form

$$v\frac{dv}{dx} + x = 0 \text{ or } \frac{dv}{dx} = -\frac{x}{v}. \tag{4}$$

The equations of isoclines for (4) are: $-x/v = k$. Constructing isoclines corresponding to various values of k we find that the phase trajectories are circles with centre at the point (0, 0) (Fig. 26).

Note that closed phase trajectories correspond to periodic motions. It can easily be seen that in the case of equa-

tion (3) we do in fact have periodic motion. Solving (3) by the methods described above we find that

$$x(t) = C_1 \cos t + C_2 \sin t.$$

Example 2. Construct phase trajectories for the equation

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} + x = 0. \tag{5}$$

Solution. We set $v = \frac{dx}{dt}$. Then equation (5) takes the form

$$\frac{dv}{dx} = \frac{v-x}{v}.$$

The equation of isoclines is $\frac{v-x}{v} = k$. The phase trajectories have the form of unwinding spirals (Fig. 27). One can perceive from the phase pattern that the motion is aperiodic, with an amplitude indefinitely increasing with time.

Construct phase trajectories for the following differential equations:

696. $\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0.$

697. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 6x = 0.$

698. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0.$

699. $\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + x = 0.$

700. $\frac{d^2x}{dt^2} - 2\left(\frac{dx}{dt}\right)^2 + \frac{dx}{dt} - 2x = 0.$

701. $\frac{d^2x}{dt^2} - x \cdot \exp\left(\frac{dx}{dt}\right) = 0 \quad (\exp u \equiv e^u).$

702. $\frac{d^2x}{dt^2} + \exp\left(-\frac{dx}{dt}\right) - x = 0.$

703. $\frac{d^2x}{dt^2} + x\left(\frac{dx}{dt}\right)^2 = 0.$

704. $\frac{d^2x}{dt^2} + (x+2)\frac{dx}{dt} = 0.$

705. $\frac{d^2x}{dt^2} - \frac{dx}{dt} + x - x^2 = 0.$

17. Boundary value problems

Consider for simplicity the second order equation

$$y'' + p_1(x)\,y' + p_2(x)\,y = 0. \tag{1}$$

Assume the coefficients $p_1(x)$ and $p_2(x)$ to be continuous in some interval (a, b). Then each solution $y(x)$ of equation (1) is defined throughout the interval. In what follows instead of equation (1) we shall consider an equation of the form

$$[p(x)\,y']' + q(x)\,y = 0, \quad p(x) > 0. \tag{2}$$

Equations (1) and (2) can be transformed into each other. Equations of the form (2) are called *self-adjoint*.

The solution of the differential equation

$$[p(x)\,y']' + q(x)\,y = 0$$

is completely determined by the initial conditions $y(x_0) = y_0$, $y'(x_0) = y'_0$. However, in many physical problems one has to seek solutions given in another way. For example, one may set the problem: find the solution of equation (2) taking prescribed values $y(a)$ and $y(b)$ at points a and b. In such cases it is common to find the values of the solution only for x in (a, b). Thus the prescribed values $y(a)$ and $y(b)$ are at the ends of the interval; therefore problems of this sort are called *boundary value problems*. In what follows we shall use as a basis the interval $(0, \pi)$ (the fundamental interval), which causes no loss of generality.

Boundary conditions of a very general form for second order equations are as follows:

$$h_0 y(0) + h_1 y'(0) = A, \quad k_0 y(\pi) + k_1 y'(\pi) = B, \tag{3}$$

where h_0, h_1, k_0, k_1, A, B are given constants, with h_0, h_1, k_0, k_1 not simultaneously zero.

If $A = B = 0$, then the boundary conditions are said to be *homogeneous*, for example:

(1) $y(0) = y(\pi) = 0.$

(2) $h_0 y(0) = y'(0), \quad y'(\pi) = -h_1 y(\pi); \quad h_0, h_1 > 0.$

(3) $y'(0) = y'(\pi) = 0,$

(4) $y(0) = y(\pi), \quad y'(0) = y'(\pi).$

Generally speaking, boundary value problems are not always solvable, i.e. sometimes there may be no solution taking the required values at the ends of the interval. For

example, the boundary value problem

$$y'' = 0, \quad y(0) - y(\pi) = 1, \quad y'(0) + y'(\pi) = 0$$

has no solution. The problem

$$y'' + \lambda y = 0, \quad y(0) = y(\pi) = 0 \tag{4}$$

has a nonzero solution only for integral values of $\sqrt{\lambda}$. It does in fact follow from the general solution of the differential equation (4)

$$y = C_1 e^{\sqrt{-\lambda} x} + C_2 e^{-\sqrt{-\lambda} x}$$

that the boundary conditions can be fulfilled if and only if $\lambda = n^2$ is the square of an integer n. The corresponding solutions are the functions $y_n = \sin nx$.

As can be seen from this example, if q in equation (2) is a function of the parameter λ, then under certain conditions there exist such values of the parameter for which the homogeneous boundary value problem for equation (2) has a nonzero solution. These values of λ are called *eigenvalues* and the corresponding solutions of the boundary value problem are called *eigenfunctions*. The latter are determined to within an arbitrary constant coefficient. Thus for the boundary value problem $y'' + \lambda y = 0$, $y(0) = y(\pi) = 0$ the numbers $1^2, 2^2, 3^2, \ldots$ and the functions $\sin x, \sin 2x, \ldots$ are the eigenvalues and eigenfunctions of the problem respectively.

Along with *simple* eigenvalues, in which case one eigenfunction corresponds (to within an arbitrary constant coefficient) to one eigenvalue, there are *multiple* eigenvalues, in which case to an eigenvalue λ_0 correspond two or more linearly independent eigenfunctions.

In solving boundary value problems (for homogeneous linear differential equations) proceed as follows: find the general solution of the given differential equation

$$y = C_1 y_1(x) + C_2 y_2(x) + \ldots + C_n y_n(x),$$

$y_1(x), y_2(x), \ldots, y_n(x)$ being linearly independent solutions. Then require that this solution $y(x)$ should satisfy the given boundary conditions. This leads to some linear system of equations which is used to determine $C_1, C_2, \ldots, C_n$. Resolving this system, if possible, one finds the solution of the given boundary value problem. Note that if there arises the problem of finding eigenvalues, the condition for the system determining $C_1, C_2, \ldots, C_n$ to have a nonzero solu-

tion is the condition determining the eigenvalues. This is generally some transcendental equation for λ.

Example 1. Solve the boundary value problem (5)

$$y'' - y = 0, \quad y'(0) = 0, \quad y(1) = 1.$$

Solution. The general solution of the given equation is

$$y(x) = C_1 e^x + C_2 e^{-x},$$

hence

$$y'(x) = C_1 e^x - C_2 e^{-x}. \tag{6}$$

Setting $x = 0$ in (6) and $x = 1$ in (5) and taking into account the boundary conditions we obtain the nonhomogeneous linear system

$$\begin{cases} C_1 - C_2 = 0, \\ C_1 e + C_2 e^{-1} = 1, \end{cases}$$

which determines the values of the constants C_1 and C_2. The determinant of the system is

$$\Delta = \begin{vmatrix} 1 & -1 \\ e & e^{-1} \end{vmatrix} = e^{-1} + e = 2\cosh 1 \neq 0,$$

consequently, it has a unique solution

$$C_1 = \frac{1}{2\cosh 1}, \quad C_2 = \frac{1}{2\cosh 1}.$$

Substituting the obtained values of C_1 and C_2 in (5) we find the solution of the given boundary value problem

$$y(x) = \frac{e^x + e^{-x}}{2\cosh 1} \text{ or } y(x) = \frac{\cosh x}{\cosh 1}.$$

Example 2. Find the eigenvalues and eigenfunctions of the following boundary value problem

$$y'' + \lambda^2 y = 0 \quad (\lambda \neq 0), \tag{7}$$

$$y'(0) = 0, \quad y(\pi) = 0. \tag{8}$$

Solution. The general solution of equation (7) is

$$y(x) = C_1 \cos \lambda x + C_2 \sin \lambda x, \tag{9}$$

whence

$$y'(x) = -C_1 \lambda \sin \lambda x + C_2 \lambda \cos \lambda x. \tag{10}$$

Setting $x = \pi$ in (9) and $x = 0$ in (10) and taking into account the boundary conditions (8) we obtain the homogeneous linear system

$$\begin{cases} C_1 \cos \lambda\pi + C_2 \sin \lambda\pi = 0, \\ \qquad\qquad\qquad C_2\lambda = 0 \end{cases} \tag{11}$$

to give C_1 and C_2.

System (11) will have nonzero solutions if and only if its determinant is zero; equating it to zero we obtain the following equation determining the eigenvalues of the given boundary value problem:

$$\begin{vmatrix} \cos \lambda\pi & \sin \lambda\pi \\ 0 & \lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda \cos \lambda\pi = 0.$$

Since according to the conditions $\lambda \neq 0$, $\cos \lambda\pi = 0$ and hence the eigenvalues are

$$\lambda = \lambda_n = \frac{2n+1}{2}, \quad n = 0, 1, 2, \ldots .$$

Corresponding to them (to within a constant coefficient C_1, which can be set equal to unity) are the eigenfunctions

$$y_n(x) = \cos \frac{2n+1}{2} x,$$

which are the solutions of the boundary value problem (7), (8).

Remark. The eigenvalues of the problems considered above form an ascending number sequence. If, however, the coefficients of a differential equation have a singular point on the boundary of the basic domain or if the basic domain is infinite, if it is the entire number line for example, then *the spectrum*, i. e. the aggregate of eigenvalues, may have a different structure. In particular, there may occur spectra containing all the numbers of an interval of the values of λ, the so-called continuous spectra. For example, suppose it is required to solve the equation $y'' + \lambda y = 0$ for the interval $-\infty < x < +\infty$ under the "boundary conditions" that $y(x)$ should be bounded at infinity. In this case any nonnegative number λ is clearly an eigenvalue with the eigenfunctions $\sin \sqrt{\lambda} x$ and $\cos \sqrt{\lambda} x$.

In solving mathematical physics problems leading to problems of finding eigenvalues one often obtains differen-

tial equations of the form

$$[p(x)y']' - q(x)\ y + \lambda\rho(x)y = 0,$$

but which are such that singularities of the differential equation may occur at the end points of the basic domain, for example the coefficient $p(x)$ may vanish. From the very nature of the problem there arise for these singular points conditions, for example, of continuity or of boundedness of the solution or of its going into infinity not higher than a given order. These conditions play the role of boundary conditions. A typical example is the Bessel equation

$$(xy')' - \frac{n^2}{x}y + \lambda xy = 0, \tag{12}$$

which appears in mathematical physics problems. Here $p(x) \equiv x$ and the above supposition that $p(x) > 0$ in the whole basic domain $0 \leqslant x \leqslant 1$ no longer holds, since $p(0) = 0$. The point $x = 0$ is a singular point for the Bessel equation.

The requirement that the solution should be bounded at this point is a special type boundary condition for the Bessel equation: find the solution of equation (12) bounded when $x = 0$ and vanishing when $x = 1$, for example.

Example 3. Solve the boundary value problem $x^2y'' + 2xy' - 6y = 0$; $y(1) = 1$; $y(x)$ being bounded when $x \to 0$.

Solution. The given equation is a Euler equation, its general solution being of the form $y(x) = C_1/x^3 + C_2x^2$ (see example 1 of Sec. 15.4). According to the conditions the solution $y(x)$ must be bounded when $x \to 0$. This requirement will be fulfilled if we set $C_1 = 0$ in the general solution. Then we shall get $y(x) = C_2x^2$. The boundary condition $y(1) = 1$ gives $C_2 = 1$. Therefore the desired solution is $y = x^2$.

706. Under what conditions has the equation $y'' + \lambda y = 0$ a nonzero solution satisfying the conditions:

(a) $y'(0) = y'(\pi) = 0$, (b) $y(0) = y(\pi)$, $y'(0) = y'(\pi)$?

707. For what values of λ has the boundary value problem $y'' + \lambda y = 0$, $y(0) = y(1) = 0$ the trivial solution $y \equiv 0$?

708. Which of the following boundary value problems is solvable:

(a) $y'' - y = 0, \quad y(0) = 0, \quad y(2\pi) = 1,$

(b) $y'' + y = 0, \quad y(0) = 0, \quad y(2\pi) = 1$?

709. Solve the boundary value problem $y'' + (\lambda - \omega^2)y = 0$, $y(0) = y(1)$, $y'(0) = y'(1)$. Consider the cases $\lambda - \omega^2 > 0$, $\lambda - \omega^2 = 0$, $\lambda - \omega^2 < 0$.

710. Find the solution of the equation $yy'' + (y')^2 + 1 = 0$, passing through the points (0, 1) and (1, 2).

Solve the following boundary value problems:

711. $y'' + y = 0, \quad y(0) = 0, y\left(\frac{\pi}{2}\right) = \alpha.$

712. $y'' - y = 0, \quad y(0) = 0, \quad y'(1) = 1.$

713. $y'' - 2y' + 2y = 0, \quad y(0) = 0, \quad y'(\pi) = e^{\pi}.$

714. $y'' + \alpha y' = 0, \quad y(0) = e^{\alpha}, \quad y'(1) = 0.$

715. $y'' + \alpha^2 y = 1, y'(0) = \alpha, \ y'(\pi) = 0 \ (0 < \alpha < 1).$

716. $y'' + y = 1, \quad y(0) = 0, \quad y'(\pi) = 0.$

717. $y'' + \lambda^2 y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0.$

718. $y'' + \lambda^2 y = 0, \quad y(0) = 0, \quad y'(\pi) = 0.$

719. $y''' + y'' - y' - y = 0, \quad y(0) = -1, \quad y'(0) = 2,$
$y(1) = 0.$

720. $y^{\text{IV}} - \lambda^4 y = 0, \quad y(0) = y''(0) = 0,$
$y(\pi) = y''(\pi) = 0.$

721. $xy'' + y' = 0, \quad y(1) = \alpha y'(1), \quad y(x)$ bounded when $x \to 0$.

722. $x^2 y^{\text{IV}} + 4xy''' + 2y'' = 0, \quad y(1) = y'(1) = 0,$
$y(x)$ bounded when $x \to 0$.

723. $x^3 y^{\text{IV}} + 6x^2 y''' + 6xy'' = 0, \quad y(1) = y'(1) = 0,$
$y(x)$ bounded when $x \to 0$.

18. Integration of differential equations in series

18.1. Expanding a solution in a power series. It is especially convenient to apply this method to linear differential equations. We shall illustrate its application using a second

order equation. Let

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{1}$$

be some given second order differential equation. Suppose that the coefficients $p(x)$ and $q(x)$ are represented by series arranged in positive integral powers of x, so that equation (1) can be rewritten as

$$y'' + (a_0 + a_1x + a_2x^2 + \ldots)\,y' + (b_0 + b_1x + b_2x^2 + \ldots)\,y = 0. \tag{2}$$

We seek the solution of this equation also in the form of the power series

$$y = \sum_{k=0}^{\infty} c_k x^k. \tag{3}$$

Substituting this expression for y and its derivatives in (2) we get

$$\sum_{k=2}^{\infty} k(k-1)\,c_k x^{k-2} + \sum_{k=0}^{\infty} a_k x^k \sum_{k=1}^{\infty} kc_k x^{k-1} + \sum_{k=0}^{\infty} b_k x^k \sum_{k=0}^{\infty} c_k x^k = 0. \tag{4}$$

Multiplying the power series, gathering together the similar terms and equating the coefficients of all powers of x in the left-hand side of (4) to zero, we obtain a series of equations:

$$\begin{array}{l|l} x^0 & 2\times 1c_2 + a_0c_1 + b_0c_0 = 0, \\ x^1 & 3\times 2c_3 + 2a_0c_2 + a_1c_1 + b_0c_1 + b_1c_0 = 0, \\ x^2 & 4\times 3c_4 + 3a_0c_3 + 2a_1c_2 + a_2c_1 + b_0c_2 + b_1c_1 + b_2c_0 = 0, \\ \cdot & \cdots\cdots\cdots\cdots\cdots\cdots \end{array} \tag{5}$$

Each subsequent equation of (5) contains one more desired coefficient. The coefficients c_0 and c_1 remain arbitrary and play the role of arbitrary constants. The first equation of (5) gives c_2, the second gives c_3, the third c_4, etc. Knowing c_0, $c_1, \ldots, c_{k+1}$, one can in general determine c_{k+2} from the $(k+1)$th equation.

In practice it is convenient to proceed as follows. Using the scheme described above determine two solutions $y_1(x)$ and $y_2(x)$, choosing $c_0 = 1$ and $c_1 = 0$ for $y_1(x)$ and $c_0 = 0$ and $c_1 = 1$ for $y_2(x)$, which is equivalent to the following

initial conditions:

$$y_1(0) - 1, \quad y_1'(0) = 0, \quad y_2(0) = 0, \quad y_2'(0) = 1.$$

Any solution of equation (1) will be a linear combination of the solutions $y_1(x)$ and $y_2(x)$.

If the initial conditions are of the form $y(0) = A$, $y'(0) = B$, then obviously

$$y = Ay_1(x) + By_2(x).$$

The following theorem holds.

Theorem. *If the series*

$$p(x) = \sum_{k=0}^{\infty} a_k x^k \text{ and } q(x) = \sum_{k=0}^{\infty} b_k x^k$$

converge when $|x| < R$, *then the power series* (3) *constructed as pointed out above also converges for these values of* x *and is a solution of equation* (1).

In particular, if $p(x)$ and $q(x)$ are polynomials of x, then the series (3) converges for any value of x.

Example 1. Find the solution of the equation

$$y'' - xy' - 2y = 0 \tag{6}$$

in the form of a power series.

Solution. We seek $y_1(k)$ in the form of the series

$$y_1(x) = \sum_{k=0}^{\infty} c_k x^k,$$

then

$$y_1'(x) = \sum_{k=1}^{\infty} kc_k x^{k-1}, \quad y_1''(x) = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}.$$

Substituting $y_1(x), y_1'(x)$, and $y_1''(x)$ in (6) we get

$$\sum_{k=2}^{\infty} (k-1) kc_k x^{k-2} - \sum_{k=1}^{\infty} kc_k x^k - 2\sum_{k=0}^{\infty} c_k x^k = 0. \tag{7}$$

Gathering together the similar terms in (7) and equating the coefficients of all powers of x to zero we obtain relations from which we find the coefficients $c_0, c_1, \ldots, c_n, \ldots$.

We set for definiteness $y_1(0) = 1$, $y_1'(0) = 0$. Then it is easy to find that

$$c_0 = 1, \quad c_1 = 0. \tag{8}$$

So we have

$$\begin{array}{c|l}
x^0 & 2c_2 - 2c_0 = 0, \text{ and so from (8) } c_2 = 1, \\
x^1 & 3 \times 2c_3 - 1 \times c_1 - 2c_1 = 0, \text{ and so from (8) } c_3 = 0, \\
x^2 & 4 \times 3c_4 - 2c_2 - 2c_2 = 0, \text{ and so } c_4 = \frac{1}{3}, \\
x^3 & 5 \times 4c_5 - 3c_3 - 2c_3 = 0, \text{ and so } c_5 = 0, \\
x^4 & 6 \times 5c_6 - 4c_4 - 2c_4 = 0, \text{ and so } c_6 = \frac{c_4}{5} = \frac{1}{3 \times 5}, \\
\cdot & \dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{array}$$

Consequently,

$$y_1(x) = 1 + x^2 + \frac{1}{3}x^4 + \frac{1}{15}x^6 + \dots. \tag{9}$$

Similarly, taking

$$y_2(x) = \sum_{k=0}^{\infty} A_k x^k \tag{10}$$

and the initial conditions $y_2(0) = 0$, $y_2'(0) = 1$, we get

$$A_0 = 0, \quad A = 1. \tag{11}$$

Substituting (10) in (6), we find

$$\sum_{k=0}^{\infty} k(k-1) A_k x^{k-2} - \sum_{k=1}^{\infty} (k+2) A_k x^k = 0,$$

$$\begin{array}{c|ll}
x^0 & 2A_2 = 0, & A_2 = 0, \\
x^1 & 3 \times 2A_3 - 3A_1 = 0, & A_3 = \frac{1}{2} \text{ (by virtue of (11))}, \\
x^2 & 4 \times 3A_4 - 4A_2 = 0, & A_4 = 0, \\
x^3 & 5 \times 4A_5 - 5A_3 = 0, & A_5 = \frac{1}{2 \times 4}, \\
x^4 & 6 \times 5A_6 - 6A_4 = 0, & A_6 = 0, \\
x^5 & 7 \times 6A_7 - 7A_5 = 0, & A_7 = \frac{1}{2 \times 4 \times 6}, \\
\cdot & \dots\dots\dots\dots\dots\dots &
\end{array}$$

It is obvious that

$$A_{2k} = 0, \quad A_{2k+1} = \frac{1}{2 \times 4 \times 6 \dots (2k)} \; (k = 1, 2, 3, \dots);$$

so

$$y_2(x) = x + \frac{x^3}{2} + \frac{x^5}{2\times 4} + \frac{x^7}{2\times 4\times 6} + \dots$$
$$= x\sum_{k=0}^{\infty} \frac{(x^2/2)^k}{k!} = xe^{x^2/2}. \qquad (12)$$

The general solution of equation (6) is of the form

$$y(x) = Ay_1(x) + By_2(x),$$

where $y_1(x)$ and $y_2(x)$ are given by formulas (9) and (12) respectively and A and B are arbitrary constants, with

$$y(0) = A, \quad y'(0) = B.$$

We shall mention another method of integrating differential equations in series found to be easier when applied to nonlinear differential equations. Suppose we are given the differential equation

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}) \qquad (13)$$

and the initial conditions

$$y|_{x=x_0} = y_0, \quad y'|_{x=x_0} = y'_0, \quad \dots, \quad y^{(n-1)}|_{x=x_0} = y_0^{(n-1)}. \qquad (14)$$

We introduce the following definition.

The function $\varphi(x)$ is said to be *holomorphic* in some neighbourhood $|x - x_0| < \rho$ of the point $x = x_0$ if it can be represented in that neighbourhood by the power series

$$\varphi(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k,$$

converging in the domain $|x - x_0| < \rho$.

Similarly the function $\varphi(x_1, x_2, \dots, x_n)$ is said to be *holomorphic over all its independent variables* in some neighbourhood

$$|x_k - x_k^{(0)}| < \rho_k \quad (k = 1, 2, \dots, n)$$

of the point $(x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$ if it can be represented by the power series

$$\varphi(x_1, x_2, \dots, x_n) = \sum c_{k_1 k_2 \dots k_n} (x_1 - x_1^{(0)})^{k_1}$$
$$\times (x_2 - x_2^{(0)})^{k_2} \dots (x_n - x_n^{(0)})^{k_n},$$

converging in the domain

$$| x_k = x_k^{(0)} | < \rho_k \quad (k = 1, 2, \ldots, n).$$

Theorem. *If the right-hand side of equation* (13) *is holomorphic over all its independent variables* $x, y, y', \ldots, y^{(n-1)}$ *in a neighbourhood* Ω:

$$| x - x_0 | < R, \quad | y - y_0 | < R_1, \quad | y' - y_0' | < R_1, \ldots, \quad | y^{(n-1)} - y_0^{(n-1)} | < R_1$$

of the point $(x_0, y_0, y_0', \ldots, y_0^{(n-1)})$, *then equation* (13) *has the unique solution*

$$y(x) = y_0 + y_0'(x - x_0) + \frac{y_0''}{2!}(x - x_0)^2 + \ldots$$

$$+ \frac{y_0^{(n-1)}}{(n-1)!}(x - x_0)^{n-1} + \sum_{k=n}^{\infty} a_k (x - x_0)^k, \quad \left(a_k = \frac{y_0^{(k)}}{k!}\right) \quad (15)$$

satisfying the initial conditions (14) *and holomorphic in some neighbourhood of the point* $x = x_0$.

Series (15) converges in the domain

$$|x - x_0| < \rho, \text{ where } \rho = a\left[1 - e^{-\frac{b}{(n+1)aM}}\right];$$

a and b being constants satisfying the conditions $0 < a < R$, $0 < b < R$ and

$$M = \max_{\Omega} |f(x, y, y', \ldots, y^{(n-1)})|.$$

The first $n + 1$ coefficients of series (15) are determined by the initial conditions (14) and the differential equation (13). The other coefficients of the series are determined by virtue of the differential equation (13) by differentiating it in succession. For example,

$$a_{n+1} = \frac{y^{(n+1)}(x_0)}{(n+1)!},$$

where

$$y^{(n+1)}|_{x=x_0} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \sum_{k=1}^{n-1} \frac{\partial f}{\partial y^{(k)}} y^{(k+1)}\right)\Bigg|_{x=x_0}$$

$$= \frac{\partial f}{\partial x}\bigg|_{x=x_0} + \frac{\partial f}{\partial y}\bigg|_{x=x_0} y_0' + \sum_{k=1}^{n-1} \frac{\partial f}{\partial y^{(k)}}\bigg|_{x=x_0} y^{(k+1)}(x_0).$$

Remark. If equation (13) is linear

$$y^{(n)} + p_1(x) y^{(n-1)} + \ldots + p_n(x) y = \psi(x),$$

where $p_k(x)$ $(k = 1, 2, \ldots, n)$ and $\psi(x)$ are functions holomorphic on the whole Ox axis, then series (15) also converges on the whole axis.

Example 2. Find the solution of the equation

$$y'' + y = 0, \tag{16}$$

satisfying the initial conditions

$$y|_{x=0} = 1, \quad y'|_{x=0} = 0. \tag{17}$$

Solution. We seek the particular solution of equation (16) satisfying the initial conditions (17) in the form of the series

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \ldots, \tag{18}$$

where $y(0) = 1$, $y'(0) = 0$.

We find from the given equation that $y''(0) = -y(0) = -1$. Differentiating both sides of equation (16) in succession and setting $x = 0$ in the equations obtained we get

$$y'''(0) = -y'(0) = 0,$$

$$y^{IV}(0) = -y''(0) = 1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y^{(n)}(0) = \begin{cases} 0, & \text{if } n = 2k-1, \\ (-1)^k, & \text{if } n = 2k \end{cases} \quad (k = 1, 2, \ldots).$$

The obtained values $y''(0)$, $y'''(0)$, ... are substituted in series (18). We obtain the desired solution in the form of the power series

$$y(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^k}{(2k)!}x^{2k} + \ldots. \tag{19}$$

It is obvious that the series contained on the right of (19) converges on the whole Ox axis to the function $y = \cos x$ which is the solution of the set initial value problem.

Example 3. Find the first four terms of the Taylor-series expansion of the solution $y = y(x)$ to the equation $y'' = e^{xy}$ satisfying the initial conditions $y|_{x=0} = 1$, $y'|_{x=0} = 0$.

Solution. It can easily be seen that the right-hand side of the equation, i.e. the function e^{xy}, can be expanded in

a power series of x and y in the neighbourhood of the point (0, 0) converging in the domain $-\infty < x < +\infty$, $-\infty < y < +\infty$ (i.e. the right-hand side is holomorphic).

We shall seek the particular solution in the form of the series

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \ldots . \qquad (20)$$

Using the equation itself we find that $y''(0) = e^{xy}\,|_{x=0} = 1$.

Differentiating both sides of the equation in succession and setting $x = 0$ in the equations obtained we get

$$y'''(0) = (y + xy')\,e^{xy}|_{x=0} = 1,$$

$$y^{IV}(0) = [2y' + xy'' + (y + xy')^2]\,e^{xy}|_{x=0} = 1.$$

Substituting the obtained values $y''(0)$, $y'''(0)$, $y^{IV}(0)$ in series (20) we get the desired expansion of the solution

$$y(x) = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots .$$

In the problems below find the first three terms of the power series expansion of the solution to the given differential equation for the specified initial conditions:

724. $y' = 1 - xy$, $\quad y|_{x=0} = 0$.

725. $y' = \dfrac{y-x}{y+x}$, $\quad y|_{x=0} = 1$.

726. $y' = \sin xy$, $\quad y|_{x=0} = 1$.

727. $y'' + xy = 0$, $\quad y|_{x=0} = 0$, $\quad y'|_{x=0} = 1$.

728. $y'' - \sin xy' = 0$, $\quad y|_{x=0} = 0$, $\quad y'|_{x=0} = 1$.

729. $xy'' + y\sin x = x$, $\quad y|_{x=\pi} = 1$, $\quad y'|_{x=\pi} = 0$.

730. $y''\ln x - \sin xy = 0$, $\quad y|_{x=e} = e^{-1}$, $\quad y'|_{x=e} = 0$.

731. $y''' + x\sin y = 0$, $\quad y|_{x=0} = \pi/2$, $\quad y'|_{x=0} = 0$, $\quad y''|_{x=0} = 0$.

Integrate in series the following differential equations:

732. $y' - 2xy = 0$, $\quad y(0) = 1$.

733. $y'' + xy' + y = 0$.

734. $y'' - xy' + y - 1 = 0$, $\quad y(0) = y'(0) = 0$.

In Problems 735 to 738 find six terms of the expansion of $y(x)$.

735. $y'' - (1 + x^2) y = 0, \quad y(0) = -2, \quad y'(0)=2.$

736. $y'' = x^2 y - y', \quad y(0) = 1, \quad y'(0) = 0.$

737. $y'' - ye^x = 0.$

738. $y' = e^y + xy, \quad y(0) = 0.$

18.2. Expanding a solution in a generalized power series. Bessel's equation. A point x_0 is said to be *an ordinary point* of the differential equation

$$y'' + p(x) y' + q(x) y = 0, \tag{21}$$

if the coefficients $p(x)$ and $q(x)$ are holomorphic at that point; otherwise it is said to be *a singular point* of the differential equation (21).

A series of the form

$$x^\rho \sum_{k=0}^{\infty} c_k x^k \quad (c_0 \neq 0), \tag{22}$$

where ρ is a given number and the power series $\sum_{k=0}^{\infty} c_k x^k$ converges in some domain $|x| < R$, is called a *generalized power series.*

If ρ is a nonnegative integral number, then the generalized power series (22) becomes an ordinary power series.

Theorem. *If the point $x = 0$ is a singular point of equation* (21) *and if the coefficients $p(x)$ and $q(x)$ of the equation can be represented as*

$$p(x) = \frac{\sum_{k=0}^{\infty} a_k x^k}{x}, \quad q(x) = \frac{\sum_{k=0}^{\infty} b_k x^k}{x^2},$$

where the series in the numerators converging in some domain $|x| < R$ and the coefficients a_0, b_0, and b_1 are not simultaneously zero, then equation (21) *has at least one solution in the form of a generalized power series*

$$y = x^\rho \sum_{k=0}^{\infty} c_k x^k \quad (c_0 \neq 0), \tag{22'}$$

which converges at least in the same domain $|x| < R$.

In order to determine the exponent ρ and the coefficients c_k, it is necessary to substitute series (22) in equation (21), cancel x^ρ and equate the coefficients of all powers of x to zero (the method of undetermined coefficients).

Here the number ρ is found from the so-called *governing equation*

$$\rho(\rho - 1) + a_0\rho + b_0 = 0, \tag{23}$$

where

$$a_0 = \lim_{x \to 0} xp(x), \qquad b_0 = \lim_{x \to 0} x^2 q(x). \tag{24}$$

Let ρ_1 and ρ_2 be roots of the governing equation (23). We shall distinguish three cases.

1. If the difference $\rho_1 - \rho_2$ is not equal to an integer or zero, then it is possible to construct two solutions of the form (22′)

$$y_1 = x^{\rho_1} \sum_{k=0}^{\infty} c_k x^k \quad (c_0 \neq 0), \qquad y_2 = x^{\rho} \sum_{k=0}^{\infty} A_k x^k \quad (A_0 \neq 0).$$

2. If the difference $\rho_1 - \rho_2$ is a positive integral number, then it is possible in general to construct only one series (the solution of equation (21))

$$y_1 = x^{\rho_1} \sum_{k=0}^{\infty} c_k x^k. \tag{25}$$

3. If equation (23) has a multiple root $\rho_1 = \rho_2$, then it is also possible to construct only one series, solution (25). In the first case the constructed solutions $y_1(x)$ and $y_2(x)$ will clearly be linearly independent (i.e. their ratio will not be a constant).

In the second and the third case we constructed only one solution for each case. Note that if the difference $\rho_1 - \rho_2$ is a positive integral number or zero, then besides solution (25) equation (21) will have a solution of the form

$$y_2 = Ay_1(x) \ln x + x^{\rho_2} \sum_{k=0}^{\infty} A_k x^k. \tag{26}$$

In this case $y_2(x)$ contains an extra term of the form $Ay_1(x) \ln x$, where $y_1(x)$ is given in the form (25).

Remark. The constant A in (26) may turn out to be zero and we shall then get an expression in the form of a generalized power series for y_2.

Example 4. Solve the equation

$$2x^2y'' + (3x - 2x^2)y' - (x + 1)\, y = 0. \tag{27}$$

Solution. Rewrite (27) in the form

$$y'' + \frac{3x - 2x^2}{2x^2} y' - \frac{x+1}{2x^2} y = 0$$

or

$$y'' + \frac{3 - 2x}{2x} y' - \frac{x+1}{2x^2} y = 0.$$

Seek the solution $y(x)$ in the form

$$y(x) = x^\rho \sum_{k=0}^{\infty} C_k x^k \qquad (C_0 \neq 0).$$

To find ρ write out the governing equation

$$\rho(\rho - 1) + a_0\rho + b_0 = 0,$$

where

$$a_0 = \lim_{x \to 0} \frac{3 - 2x}{2} = \frac{3}{2}, \qquad b_0 = \lim_{x \to 0} \left(-\frac{x+1}{2} \right) = -\frac{1}{2},$$

i. e. $\rho(\rho - 1) + \frac{3}{2}\rho - \frac{1}{2} = 0$ or

$\rho^2 + \frac{1}{2}\rho - \frac{1}{2} = 0$; so $\rho_1 = 1/2$, $\rho_2 = -1$.

According to the mentioned rule, take

$$y_1(x) = x^{\frac{1}{2}} \sum_{k=0}^{\infty} C_k x^k, \quad (x > 0); \quad y_2(x) = \frac{1}{x} \sum_{k=0}^{\infty} A_k x^k.$$

To find $C_0, C_1, \ldots, C_n$, it is necessary to substitute $y_1(x)$ and its derivatives $y_1'(x)$ and $y_1''(x)$ in equation (27):

$$y_1(x) = \sum_{k=0}^{\infty} C_k x^{k+\frac{1}{2}}, \quad y_1'(x) = \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) C_k x^{k-\frac{1}{2}},$$

$$y_1''(x) = \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right)\left(k - \frac{1}{2}\right) C_k x^{k-\frac{3}{2}}.$$

Substituting $y_1(x)$, $y_1'(x)$, $y_1''(x)$ in (27) gives

$$2x^2\sum_{k=0}^{\infty}\left(k^2-\frac{1}{4}\right)C_kx^{k-\frac{3}{2}} + (3x-2x^2)\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)C_kx^{k-\frac{1}{2}} - (x+1)\sum_{k=0}^{\infty}C_kx^{k+\frac{1}{2}}=0. \qquad (28)$$

After transformations (28) is rewritten as

$$\sum_{k=0}^{\infty}k(2k+3)\,C_kx^{k+1/2}-\sum_{=0}^{\infty}2\,(k+1)\,C_kx^{k+3/2}=0. \qquad (29)$$

Since we seek a solution for $x>0$, it is possible to cancel $x^{1/2}$ from (29), which gives

$$\sum_{k=2}^{\infty}k(2k+3)\,C_kx^k-\sum_{k=0}^{\infty}2\,(k+1)\,C_kx^{k+1}=0. \qquad (30)$$

From this find relations to determine the coefficients

$$\left.\begin{array}{l|l} x^1 & 1\times 5C_1-2\times 1C_0=0,\\ x^2 & 2\times 7C_2-2\times 2C_1=0,\\ x_3 & 3\times 9C_3-2\times 3C_2=0,\\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ x^n & n(2n+3)\,C_n-2nC_{n-1}=0,\\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right. \qquad (31)$$

Setting $C_0=1$ in the first equation of relations (31) we get $C_1=2/5$. From the second equation we have $C_2=\frac{2^2}{5\times 7}$, from the third we have $C_3=\frac{2^3}{5\times 7\times 9}$, etc. It can easily be seen that

$$C_n=\frac{2^n}{5\times 7\times 9\ldots(2n+3)} \quad (n=1, 2, 3, \ldots);$$

so

$$y_1(x)=x^{\frac{1}{2}}\left[1+\sum_{k=1}^{\infty}\frac{(2x)^k}{5\times 7\times 9\ldots(2k+3)}\right]. \qquad (32)$$

Similarly find the coefficients A_k. It turns out that for $A_0 = 1$

$$A_1 = 1, \quad A_2 = \frac{1}{2!}, \ \ldots, \quad A_k = \frac{1}{k!},$$

so that

$$y_2(x) = \frac{1}{x} \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{or} \quad y_2(x) = \frac{e^x}{x}. \tag{33}$$

The general solution of equation (27) is $y(x) = Ay_1(x) + By_2(x)$, where A and B are arbitrary constants and the functions $y_1(x)$ and $y_2(x)$ are given by formulas (32) and (33).

Example 5. The interaction of two nuclei can be described to a good approximation using the potential of mesonic forces $V = Ae^{-\alpha x}/x$ ($A < 0$ corresponding to attraction). Find in the form of a series the solution of the Schrödinger equation

$$y'' + k\left(E - \frac{Ae^{-\alpha x}}{x}\right) y = 0, \tag{34}$$

where α, A, E, and $k = 2m/\hbar$ are constants (restrict yourself to three nonzero coefficients of the series corresponding to the larger root of the governing equation).

Solution. We seek the solution $y(x)$ of the given equation in the form of a generalized power series

$$y(x) = x^\rho \sum_{k=0}^{\infty} c_k x^k.$$

The coefficients A_0 and B_0 of the governing equation $\rho(\rho - 1) + A_0\rho + B_0 = 0$ are

$$A_0 = \lim_{x \to 0} xp(x) = 0, \quad \text{since} \quad p(x) \equiv 0,$$

$$B_0 = \lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 k\left(E - \frac{Ae^{-\alpha x}}{x}\right)$$

$$= \lim_{x \to 0} k(Ex^2 - Axe^{-\alpha x}) = 0,$$

so it takes the form $\rho(\rho - 1) = 0$, whence $\rho_1 = 1$, $\rho_2 = 0$. The generalized power series for the case $\rho = 1$ is

$$y(x) = x \sum_{k=0}^{\infty} c_k x^k = c_0 x + c_1 x^2 + c_2 x^3 + c_3 x^4 + \ldots; \tag{35}$$

then

$$y' = c_0 + 2c_1x + 3c_2x^2 + 4c_3x^3 + \ldots,$$
$$y'' = 2c_1 + 6c_2x + 12c_3x^3 + \ldots.$$

In addition we have

$$e^{-\alpha x} = 1 - \alpha x + \frac{\alpha^2x^2}{2!} - \frac{\alpha^3x^3}{3!} + \frac{\alpha^4x^4}{4!} - \ldots .$$

We substitute the series for y'', y, and $e^{-\alpha x}$ in equation (14):

$$2c_1 + 6c_2x + 12c_3x^2 + \ldots + \left[kE - kA\left(\frac{1}{x} - \alpha + \frac{\alpha^2x}{2!} - \frac{\alpha^3x^2}{3!} + \ldots\right)\right](c_0x + c_1x^2 + c_2x^3 + \ldots) = 0.$$

We equate the coefficients of all powers of x to zero:

$$\begin{array}{l|l} x^0 & 2c_1 - kAc_0 = 0, \\ x^1 & 6c_2 + (kE + \alpha)\, c_0 - kAc_1 = 0, \\ \cdot & \ldots\ldots\ldots\ldots\ldots \end{array}$$

From the obtained equations we find in succession

$$c_1 = \frac{Ak}{2} c_0, \quad c_2 = \frac{Akc_1 - (kE + \alpha A)\, c_0}{6}$$

or

$$c_2 = \frac{1}{6}\left(\frac{A^2k^2}{2} - kE - \alpha kA\right) c_0, \text{ etc.}$$

Substituting the obtained values of the coefficients in series (35) we get

$$y(x) = c_0x\left[1 + \frac{Ak}{2}x + \frac{1}{6}\left(\frac{A^2k^2}{2} - kE - \alpha kA\right)x^2 + \ldots\right],$$

c_0 being an arbitrary constant.

Integrate in series the following differential equations

739. $4xy'' + 2y' + y = 0.$

740. $(1 + x)\, y' - ny = 0.$

741. $9x(1 - x)\, y'' - 12y' + 4y = 0.$

742. A quantum analysis of the Stark effect (using a parabolic system of coordinates) leads to the differential equ-

ation

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy') + \left(\frac{1}{2}Ex + \alpha - \frac{m^2}{4x} - \frac{1}{4}Fx^2\right) y = 0,$$

α, E, F, m being constants. Using the largest of the roots of the governing equation, obtain a solution in the neighbourhood of the point $x = 0$ (find the first three coefficients).

743. In case of the absence of azimuthal dependence a quantum mechanical consideration of the hydrogen molecule ion leads to the equation

$$\frac{d}{dx}[(1-x^2)y'] + (\alpha + \beta x^2) y = 0,$$

α, β being constants. Find a solution of this equation in the form of a series (compute the first three nonzero coefficients of the expansion).

Example 6. Solve the Bessel equation

$$x^2y'' + xy' + (x^2 - p^2) y = 0, \quad x > 0, \tag{36}$$

p being a given constant.

Solution. We rewrite (36) as

$$y'' + \frac{1}{x}y' + \frac{x^2 - p^2}{x^2} y = 0.$$

Here $p(x) = \frac{1}{x}$, $q(x) = \frac{x^2 - p^2}{x^2}$, so that

$$a_0 = \lim_{x\to 0} xp(x) = 1, \quad b_0 = \lim_{x \to 0} x^2 q(x) = -p^2$$

(see formulas (24)). The governing equation for ρ is

$$\rho(\rho - 1) + 1\cdot\rho - p^2 = 0 \text{ or } \rho^2 - p^2 = 0,$$

whence $\rho_1 = p$, $\rho_2 = -\rho$.

We seek the first particular solution of the Bessel equation (36) in the form of the generalized power series $y = x^p \sum_{k=0}^{\infty} C_k x^k$. Substituting y, y', and y'' in equation (36) we get

$$x^2 \sum_{k=0}^{\infty} C_k (k+p)(k+p-1) x^{k+p-2}$$
$$+ x \sum_{k=0}^{\infty} C_k (k+p) x^{k+p-1} + (x^2 - p^2) \sum_{k=0}^{\infty} C_k x^{k+p} = 0$$

or, after some simple transformations and cancelling x^p,

$$\sum_{k=0}^{\infty} [(k+p)^2 - p^2] C_k x^k + \sum_{k=0}^{\infty} C_k x^{k+2} = 0.$$

From this, equating to zero the coefficients of all powers of x, we have

$$\begin{array}{l|l}
x^0 & (p^2 - p^2) C_0 = 0, \\
x^1 & [(1+p)^2 - p^2] C_1 = 0, \\
x^2 & [(2+p)^2 - p^2] C_2 + C_0 = 0, \\
x^3 & [(3+p)^2 = p^2] C_3 + C_1 = 0, \\
x^4 & [(4+p)^2 - p^2] C_4 + C_2 = 0, \\
\cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
x^k & [(k+p)^2 - p^2] C_k + C_{k-2} = 0, \\
\cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array} \tag{37}$$

The first of relations (37) is satisfied for any value of the coefficient C_0. The second of relations (37) gives $C_1 = 0$, the third gives $C_2 = -\frac{C_0}{(2+p)^2 - p^2} = -\frac{C_0}{2^2(1+p)}$, the fourth $C_3 = 0$, the fifth

$$C_4 = -\frac{C_2}{(4+p)^2 - p^2} = \frac{C_0}{2^4 (1+p)(2+p) \times 1 \times 2}.$$

It is obvious that all the coefficients with odd indices are zero: $C_{2k+1} = 0$, $k = 0, 1, 2, \ldots$. The coefficients with even indices are of the form

$$C_{2k} = \frac{(-1)^k C_0}{2^{2k} (p+1)(p+2) \ldots (p+k)\, k!}, \quad k = 1, 2, \ldots .$$

To simplify further computations assume

$$C_0 = \frac{1}{2^p \Gamma(p+1)}, \tag{38}$$

$\Gamma(\nu)$ being the Euler gamma function. *The Euler gamma function* $\Gamma(\nu)$ is defined for all positive values (as well as for all complex values with a positive real part) as follows:

$$\Gamma(\nu) = \int_0^{\infty} e^{-x} x^{\nu-1}\, dx.$$

The gamma function possesses the following important properties:

1. $\Gamma(\nu+1)=\nu\Gamma(\nu)$.
2. $\Gamma(1)=1$.

If k is a positive integral number, then

3. $\Gamma(\nu+k+1)=(\nu+1)(\nu+2)\ldots(\nu+k)\Gamma(\nu+1)$.
4. $\Gamma(k+1)=k!$

Using (38) and the properties of the gamma function we proceed to transform the coefficient C_{2k}:

$$C_{2k}=\frac{(-1)^k}{2^{2k}(p+1)(p+2)\ldots(p+k)\times k!\times 2^p\Gamma(p+1)}=$$

$$=\frac{(-1)^k}{2^{2k+p}\times k!\,\Gamma(p+k+1)},$$

for $(p+1)(p+2)\ldots(p+k)\,\Gamma(p+1)$ is, according to property 3, $\Gamma(p+k+1)$. The particular solution of the Bessel equation to be denoted by J_p below now takes the form

$$J_p(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(p+k+1)}\left(\frac{x}{2}\right)^{2k+p}. \quad (39)$$

It is customary to call this function *the Bessel function of the first kind and order p*.

The second particular solution of the Bessel equation (36) is sought in the form

$$y=x^{-p}\sum_{k=0}^{\infty}C_kx^k,$$

p being a second root of the governing equation. It is clear that this solution can be derived from solution (39) by replacing p by $-p$, since equation (36) contains p to the even power and remains unchanged when p is replaced by $-p$.

So

$$J_{-p}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(k+1-p)}\left(\frac{x}{2}\right)^{2k-p}.$$

This function is called *the Bessel function of the first kind and order $-p$*.

If p *is not* integer, then the solutions $J_p(x)$ and $J_{-p}(x)$ are linearly independent, since their expansion begins with different powers of x and so the linear combination $\alpha_1 J_p(x) + \alpha_2 J_{-p}(x)$ may be identically zero only for $\alpha_1 = \alpha_2 = 0$.

If p is an integer, then it is possible to establish linear dependence of the functions $J_p(x)$ and $J_{-p}(x)$, namely it turns out that

$$J_{-n}(x) = (-1)^n J_n(x) \quad (n \text{ being an integer}).$$

So for an integral p it is necessary to seek another solution instead of $J_{-p}(x)$, a solution that would be linearly independent of $J_p(x)$. To this end we introduce a new function

$$Y_p(x) = \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi}, \tag{40}$$

at first assuming p to be a noninteger.

It is obvious that the function $Y_p(x)$ thus defined is a solution of equation (36) (due to the fact that it represents a linear combination of the particular solutions $J_p(x)$ and $J_{-p}(x)$).

Proceeding to the limit in (40) when p *tends to an integer* we obtain the particular solution $Y_p(x)$ linearly independent of $J_p(x)$ and already defined for integral values of p.

The function $Y_p(x)$ introduced here is called *the Bessel function of the second kind and order p*. Thus we have constructed *the fundamental system of solutions* of the Bessel equation (36) for any p, integral or nonintegral. It follows that the general solution of equation (36) can be represented as

$$y = AJ_p(x) + BY_p(x),$$

A and B being arbitrary constants.

If p is not an integer, the general solution of the Bessel equation may be taken in the form

$$y = C_1 J_p(x) + C_2 J_{-p}(x),$$

where C_1 and C_2 are arbitrary constants.

Remark 1. The frequently occurring equation

$$x^2 y'' + xy' + (k^2 x^2 - p^2)\, y = 0, \tag{41}$$

k being some constant ($k \neq 0$), can be reduced to the Bessel equation

$$\xi^2 \frac{d^2y}{d\xi^2} + \xi \frac{dy}{d\xi} + (\xi^2 - p^2)\, y = 0 \tag{42}$$

by substituting $\xi = kx$.

The general solution of equation (42) (when p is non-integral) will be

$$y = C_1 J_p(\xi) + C_2 J_{-p}(\xi)$$

and then the general solution of equation (41) takes the form

$$y = C_1 J_p(kx) + C_2 J_{-p}(kx).$$

When p is integral $y = C_1 J_p(kx) + C_2 Y_p(kx)$.

Remark 2. A large class of equations of the form

$$x^2 \frac{d^2y}{dx^2} + ax \frac{dy}{dx} + (b + cx^m)\, y = 0, \tag{43}$$

a, b, c, m being constants ($c > 0$, $m \neq 0$), can be reduced, by introducing a new variable t and a new function u by the formulas

$$y = \left(\frac{t}{\gamma}\right)^{-\alpha/\beta} u, \quad x = \left(\frac{t}{\gamma}\right)^{1/\beta}$$

to the Bessel equation

$$t^2 \frac{d^2u}{dt^2} + t \frac{du}{dt} + (t^2 - p^2)\, u = 0,$$

where $\alpha = \frac{a-1}{2}$, $\beta = \frac{m}{2}$, $\gamma = \frac{2\sqrt{c}}{m}$, $p^2 = \frac{(a-1)^2 - 4b}{m^2}$. When $c = 0$ or $m = 0$ equation (43) is a Euler equation.

Example 7. Reduce the equation

$$x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + (x^4 - 12)\, y = 0 \tag{44}$$

to the Bessel equation and find its general solution.

Solution. In our case the coefficients are $a = -3$, $b = -12$, $c = 1$, $m = 4$, therefore

$$\alpha = \frac{a-1}{2} = -2, \quad \beta = \frac{m}{2} = 2, \quad -\frac{\alpha}{\beta} = 1, \quad \frac{1}{\beta} = \frac{1}{2},$$

$$\gamma = \frac{2\sqrt{c}}{m} = \frac{1}{2}, \quad p^2 = \frac{(a-1)^2 - 4b}{m^2} = 4.$$

We introduce a new independent variable t and a function u by the formulas

$$y = \left(\frac{t}{\gamma}\right)^{-\alpha/\beta} u \quad \text{or} \quad y = 2ut, \quad \text{where} \quad u = u(t), \tag{45}$$

$$x = \left(\frac{t}{\gamma}\right)^{1/\beta} \quad \text{or} \quad x = \sqrt{2t}; \tag{46}$$

then

$$\frac{dy}{dx} = \frac{\frac{d}{dt}(2ut)}{\frac{dx}{dt}} = \sqrt{2t}\frac{d}{dt}(2ut) = 2\sqrt{2}\left(t^{3/2}\frac{du}{dt} + t^{1/2}u\right).$$

Similarly we find that

$$\frac{d^2y}{dx^2} = 4t^2\frac{d^2u}{dt^2} + 10t\frac{du}{dt} + 2u.$$

Substituting in (44) the expressions for x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in terms of t and u we obtain the Bessel equation

$$t^2\frac{d^2u}{dt^2} + t\frac{du}{dt} + (t^2 - 4)u = 0,$$

whose general solution is

$$u = C_1J_2(t) + C_2Y_2(t).$$

Changing to the variables x and y by the formulas $t = x^2/2$, $u = y/x^2$ derived from (45) and (46) we obtain the general solution of the given equation

$$y = x^2\left[C_1J_2\left(\frac{x^2}{2}\right) + C_2Y_2\left(\frac{x^2}{2}\right)\right].$$

Find the general solutions of the following Bessel equations:

744. $x^2y'' + xy' + \left(4x^2 - \frac{1}{9}\right)y = 0.$

745. $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0.$

746. $y'' + \frac{1}{x}y' + \frac{1}{9}y = 0.$

747. $y'' + \frac{1}{x}y' + 4y = 0.$

748. $x^2y'' - 2xy' + 4(x^4 - 1)y = 0.$

749. $xy'' + \frac{1}{2}y' + \frac{1}{4}y = 0.$

750. $y'' + \frac{5}{x}y' + y = 0.$

751. $y'' + \frac{3}{x}y' + 4y = 0.$

18.3. Finding periodic solutions of linear differential equations. Let the second order nonhomogeneous linear differential equation with constant coefficients

$$y'' + p_1y' + p_2y = f(x) \tag{47}$$

be given, $f(x)$ being a periodic function of period 2π that can be expanded into a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{48}$$

The periodic solution of equation (47) is sought in the form

$$y(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_1 \cos nx + B_n \sin nx). \tag{49}$$

We substitute series (49) in equation (47) and select its coefficients so that equation (47) is satisfied formally. Equating the left- and right-hand free terms and coefficients of $\cos nx$ and $\sin nx$ of the obtained equation we get

$$A_0 = \frac{a_0}{p_2}; \quad A_n = \frac{(p_2 - n^2)a_n - p_1nb_n}{(p_2 - n^2)^2 + p_1^2n^2};$$
$$B_n = \frac{(p_2 - n^2)b_n + p_1na_n}{(p_2 - n^2)^2 + p_1^2n^2}, \quad n - 1, 2, \ldots \tag{50}$$

The first of equations (50) gives the necessary condition for the existence of a solution of the form (49): if $a_0 \neq 0$, then it is necessary that $p_2 \neq 0$. Substituting (50) in (49) we find that

$$y(x) = \frac{a_0}{2p_2} +$$
$$+ \sum_{n=1}^{\infty} \frac{[(p_2 - n^2)a_n - p_1nb_n] \cos nx + [(p_2 - n^2)b_n + p_1na_n] \sin nx}{(p_2 - n^2)^2 + p_1^2n^2}. \tag{51}$$

When $p_1 = 0$ and $p_2 = n^2$, with $n = 1, 2, \ldots$, a periodic solution will exist provided that

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx = 0, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx = 0. \tag{52}$$

For $k \neq n$ the coefficients A_k and B_k are found from formulas (50) and the coefficients A_n and B_n remain arbitrary, since the expression $A_n \cos nx + B_n \sin nx$ is the general solution of the corresponding homogeneous equation.

If conditions (52) fail to hold, equation (47) has no periodic solutions (there arises resonance). When $p_2 = 0$ and $a_0 = 0$ the coefficient A_0 remains undetermined and equation (47) has an infinite number of periodic solutions differing from one another by a constant term.

If the right-hand side $f(x)$ of equation (47) is of period $2l \neq 2\pi$, then it is necessary to expand $f(x)$ using period $2l$ and seek a solution of equation (47) in the form

$$y = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l} \right),$$

formulas (50) changing accordingly.

Example 8. Find periodic solutions of the equation

$$y'' + 4y = \sum_{n=3}^{\infty} \frac{\sin nx}{n^2}.$$

Solution. We have $p_1 = 0$, $p_2 = 4 = 2^2$, $a_0 = 0$, $a_n = 0$, $b_n = 1/n^2$ $(n = 3, 4, \ldots)$. The function

$$f(x) = \sum_{n=3}^{\infty} \frac{\sin nx}{n^2}$$

does not contain the resonance term $a_2 \cos 2x + b_2 \sin 2x$, so the equation has periodic solutions, an infinite number of them.

Using formulas (50) we find the coefficients

$$A_0 = A_n = 0, \quad B_1 = 0, \quad B_n = \frac{1}{n^2 (4 - n^2)}, \quad n = 3, 4, \ldots .$$

All periodic solutions are given by the formula

$$y(x) = A_2 \cos 2x + B_2 \sin 2x - \sum_{n=3}^{\infty} \frac{\sin nx}{n^2(n^2-4)},$$

A_2 and B_2 being arbitrary constants.

Example 9. Find periodic solutions of the equation $y'' + y = \cos x$.

Solution. In this case $p_1 = 0$, $p_2 = 1$. We examine if conditions (52) hold. We have

$$\int_0^{2\pi} \cos x \cos x \, dx = \int_0^{2\pi} \cos^2 x \, dx = \pi \neq 0;$$

$$\int_0^{2\pi} \cos x \sin x \, dx = 0 \quad (\text{here } n = 1).$$

Conditions (52) for the existence of a periodic solution fail to hold. Consequently, the given equation has no periodic solutions. Indeed, the general solution of the equation $y'' + y = \cos x$ is

$$y(x) = C_1 \cos x + C_2 \sin x + \frac{1}{2} x \sin x,$$

it is clearly not periodic because of the term $\frac{1}{2} x \sin x$ being present in it.

Example 10. Find a periodic solution of the equation $y'' - y = |\sin x|$.

Solution. The function $f(x) = |\sin x|$ is periodic of period π. We expand it into the Fourier series in the interval $(-\pi, \pi)$:

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1} \; (-\pi, \pi).$$

We seek a solution of the given equation in the form

$$y(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx).$$

We have

$$p_1 = 0, \quad p_2 = -1; \quad a_0 = 4/\pi, \quad a_{2n-1} = 0,$$

$$a_{2n} = -\frac{4}{\pi} \frac{1}{4n^2-1}, \quad b_n = 0 \quad (n = 1, 2, \ldots).$$

Formulas (50) give

$$A_0 = -\frac{4}{\pi}, \quad A_{2n-1} = 0, \quad A_{2n} = -\frac{4}{\pi}\frac{1}{16n^2-1}, \quad B_n = 0.$$

So the equation has a periodic solution of the form

$$y(x) = -\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{16n^2-1}.$$

Find periodic solutions (if there are any) of the following differential equations:

752. $y'' + 3y = 1 + \sum\limits_{n=1}^{\infty}\frac{\cos nx + \sin nx}{n^3}$.

753. $y'' + y = \sum\limits_{n=1}^{\infty}\frac{\cos nx}{n^3}$.

754. $y'' + y = \sum\limits_{n=2}^{\infty}\frac{\cos nx}{n^2}$.

755. $y'' + y = \cos x \cos 2x$.

756. $y'' + y' = \sum\limits_{n=1}^{\infty}\frac{\sin nx}{n^2}$.

757. $y'' + 4y = \cos^2 x$.

758. $y'' - 4y' + 4y = \pi^2 - x^2, \; -\pi < x < \pi$.

759. $y'' - 4y = |\cos \pi x|$.

760. $y'' - 4y' + 4y = \arcsin(\sin x)$.

761. $y'' + 9y = \sin^3 x$.

18.4. Asymptotic integration. Let

$$A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \ldots + \frac{A_n}{x^n} + \ldots \tag{53}$$

be a series (possibly divergent). Denote by $S_n(x)$ the sum of the first $n+1$ terms of the series.

Series (53) will be said to be *an asymptotic expansion* of the function $f(x)$ for sufficiently large $|x|$, if the expression

$$R_n(x) = x^n\{f(x) - S_n(x)\}$$

satisfies the condition

$$\lim_{|x|\to 0} R_n(x) = 0 \quad \text{or} \quad R_n(x) = o\left(\frac{1}{x_n}\right), \tag{54}$$

(n being any fixed number) even if $\lim_{n\to\infty} |R_n(x)| = \infty$ (x being fixed). The fact that the given series is an asymptotic expansion of the function $f(x)$ (an asymptotic power series) is designated thus

$$f(x) \sim \sum_{n=0}^{\infty} A_n x^{-n}.$$

The significance of asymptotic expansion lies in the fact that series (53) can serve as a source of approximation formulas

$$f(x) \approx A_0 + \frac{A_1}{x} + \ldots + \frac{A_n}{x^n}, \quad n = 0, 1, 2, \ldots,$$

so the difference $f(x) - S_n(x) = \rho_n(x)$ for $|x| \to \infty$ will be an infinitesimal of order higher than n, i.e.

$$\lim_{|x|\to\infty} \frac{\rho_n(x)}{\frac{1}{|x|^n}} = 0 \quad \text{or} \quad \lim_{|x|\to\infty} |x|^n \rho_n(x) = 0.$$

Example 11. Consider the function

$$f(x) = \int_x^{+\infty} t^{-1} e^{x-t}\, dt, \quad x > 0. \tag{55}$$

Applying n times integration by parts we get

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \ldots + (-1)^{n-1} \frac{(n+1)!}{x^n} + \\ + (-1)^n n! \int_x^{+\infty} \frac{e^{x-t}}{t^{n+1}}\, dt.$$

Denote $u_{n-1} = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$ and set

$$S_n(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \ldots + \frac{(-1)^n n!}{x^{n+1}} = \sum_{m=0}^{n} u_m.$$

We have $\left|\frac{u_m}{u_{m-1}}\right| = \frac{m}{x} \to \infty$ for $m \to \infty$, so the series $\sum_{m=0}^{\infty} u_m$ is convergent for all values of x. However, this series can be applied to compute the values of $f(x)$ for large values of x. Indeed, fix some value n:

$$f(x) - S_n(x) = (-1)^{n+1}(n+1)! \int_x^{+\infty} \frac{e^{x-t}}{t^{n+2}}\, dt$$

from this, with $e^{x-t} \leqslant 1$ $(t \geqslant x)$, we have

$$|f(x) - S_n(x)| = (n+1)! \int_x^{+\infty} \frac{e^{x-t}}{t^{n+2}}\, dt$$

$$\leqslant (n+1)! \int_x^{+\infty} \frac{dt}{t^{n+2}} = \frac{n!}{x^{n+1}}. \qquad (56)$$

The right-hand side of this inequality may be made arbitrarily small for values of x sufficiently small. Thus for $x \geqslant 2n$ we have

$$|f(x) - S_n(x)| < \frac{1}{2^{n+1} n^2},$$

therefore the value of the function $f(x)$ can be computed to a great accuracy for large values of x if we take a suitable number of terms of the series $\sum_{m=0}^{\infty} u_m$. It follows from estimate (56) that

$$R_n(x) = x^n \{f(x) - S_n(x)\} \to 0 \text{ as } x \to \infty$$

for any fixed n, so the series $\sum_{m=0}^{\infty} u_m$ gives an asymptotic expansion of the given function $f(x)$.

If condition (54) holds, then for the coefficients A_k of series (53) we get from (54)

$$A_0 = \lim_{x \to \infty} f(x), \quad A_n = \lim_{x \to \infty} x^n \{f(x) - S_{n-1}(x)\},$$

$$n = 1, 2, \ldots. \qquad (57)$$

It follows that if the function $f(x)$ has an asymptotic power series, this latter is unique.

On the contrary, one and the same series of the form (53) may serve as an asymptotic power series for different functions. For example, for the function

$$f(x) = e^{-x} \quad (0 < x < +\infty)$$

an asymptotic power series, by virtue of (57), is series (53) all the coefficients A_n of which are zero: $A_n = 0$ $(n = 0, 1, \ldots)$. It is obvious that the same series is an asymptotic power series for the function $f(x) \equiv 0$ as well. It is said that an asymptotic power series represents a class of asymptotically equal functions rather than a single function.

Operations on asymptotic series.

1. If

$$f(x) \sim \sum_{k=0}^{\infty} A_k x^{-k}, \quad g(x) \sim \sum_{k=0}^{\infty} B_k x^{-k}, \tag{58}$$

then

$$f(x) \pm g(x) \sim \sum_{k=0}^{\infty} (A_k \pm B_k) x^{-k}. \tag{59}$$

2. If the asymptotic power series (58) occur, then an asymptotic expansion of the function $f(x)\,g(x)$ can be obtained by multiplying formally series (58).

3. If the function $f(x)$ has an asymptotic power series

$$f(x) \sim \sum_{k=2}^{\infty} \frac{A_k}{x^k} \tag{60}$$

beginning with the term x^{-2}, then we have the asymptotic power series

$$\int_x^{\infty} f(x)\,\mathrm{d}x \sim \sum_{k=2}^{\infty} \int_x^{\infty} \frac{A_k}{x^k}\,\mathrm{d}x$$

or

$$\int_x^{\infty} f(x)\,\mathrm{d}x \sim \sum_{k=2}^{\infty} \frac{A_k}{(k-1)\,x^{k-1}}, \tag{61}$$

i.e. the asymptotic power series (60) can be integrated formally term by term.

4. Formal term-by-term differentiation of an asymptotic power series is in general inadmissible.

Consider indeed the function

$$f(x) = e^{-x} \sin e^x, \quad 0 < x < +\infty.$$

Its asymptotic power series is a series with the coefficients $A_k = 0$, $k = 0, 1, \ldots$, while the derivative of the function $f'(x) = -e^{-x} \sin e^x + \cos e^x$ has no asymptotic power series, since $f'(x)$ has even no limit when $x \to +\infty$.

If, however, the function $f(x) \sim A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \ldots$ is differentiable and the function $f'(x)$ can be expanded into an asymptotic power series, then

$$f'(x) \sim -\frac{A_1}{x^2} - \frac{2A_2}{x^3} - \frac{3A_3}{x^4} - \ldots.$$

762. Show that

$$\int_0^{+\infty} \frac{e^{-xt}}{1+t^2}\,dt \sim \frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} - \ldots,$$

$x > 0$ being real.

763. Show that

$$e^z z^{-a} \int_z^{\infty} e^{-x} x^{a-1}\,dx \sim \frac{1}{z} + \frac{a-1}{z^2} + \frac{(a-1)(a-2)}{z^3} + \ldots$$

for large positive values of z.

764. Find the asymptotic series of the function

$$f(x) = \frac{1}{x} + e^{-x} \sin e^{2x}, \quad 0 < x < \infty.$$

Show that the derivative $f'(x)$ has no asymptotic power series.

Applications to the integration of differential equations.

Example 12. Consider the differential equation

$$\frac{dy}{dx} + y = \frac{1}{x}. \tag{62}$$

The series

$$\frac{1}{x} + \frac{1!}{x^2} + \frac{2!}{x^3} + \ldots + \frac{n!}{x^{n+1}} + \ldots \tag{63}$$

convergent for all values of x, satisfies the given equation formally, which is easily shown by a direct check. Equation

(62) is satisfied by the function*

$$y = e^{-x} \int_{-\infty}^{x} t^{-1} e^{t}\, dt,$$

the right-hand integral converging when $x < 0$. By repeated integration by parts we get

$$e^{-x} \int_{-\infty}^{x} t^{-1} e^{t}\, dt = \frac{1}{x} + \frac{1!}{x^2} + \frac{2!}{x^3} + \ldots + \frac{n!}{x^{n+1}} + \rho_n,$$

where

$$\rho_n = (n+1)!\, e^{-x} \int_{-\infty}^{x} \frac{e^{t}}{t^{n+2}}\, dt.$$

When $x < 0$ we have

$$|\rho_n| \leqslant (n+1)!\, e^{-x} \frac{1}{|x|^{n+2}} \int_{-\infty}^{x} e^{t}\, dt = \frac{(n+1)!}{|x|^{n+2}}.$$

Consequently, taking the first n terms of the series leads to an error smaller than the $(n+1)$th term. It can easily be seen that in this case

$$R_n(x) = x^n \{f(x) - S_n(x)\} = x^n \rho_n(x) \to 0 \text{ as } x \to -\infty.$$

Therefore the constructed series is asymptotic and can be used to compute the integral and, by the same token, the solution of equation (62).

Example 13. If $J_\nu(x)$ is a solution of the Bessel equation $x^2 y'' + xy' + (x^2 - \nu^2)\, y = 0$, then, on substituting the function $x^{-1/2} y(x)$ for J_ν, we shall discover that $y(x)$ satisfies the equation

$$y'' + \left(1 - \frac{\nu^2 - \frac{1}{4}}{x^2}\right) y = 0. \tag{64}$$

For large x ($x \gg \nu$) it is natural that we should try to replace this equation by the equation

$$y_1'' + y_1 = 0, \tag{65}$$

---

* The function defined by the integral $\int_{-\infty}^{x} \frac{e^t}{t}\, dt$ is called *an exponential integral function* and is denoted $Ei(x)$.

which has a solution

$$y_1 = a_0 \sin x + b_0 \cos x.$$

It is possible to improve the accuracy (for large x) by replacing the constants a_0 and b_0 by series in terms of negative powers of x:

$$\sum_{n=0}^{\infty} a_n x^{-n}, \quad \sum_{n=0}^{\infty} b_n x^{-n}.$$

This means that a solution of equation (64) may be sought in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{-n} \sin x + \sum_{n=0}^{\infty} b_n x^{-n} \cos x. \qquad (66)$$

On substituting expression (66) in equation (64) we get

$$y(x) = \left[a_0 - \frac{\nu^2 - \frac{1}{4}}{2x} b_0 - \frac{\left(\nu^2 - \frac{1}{4}\right)\left(\nu^2 - \frac{9}{4}\right)}{2(2x)^2} a_0 + \ldots\right] \times \sin x + \left[b_0 - \frac{\nu^2 - \frac{1}{4}}{2x} a_0 - \frac{\left(\nu^2 - \frac{1}{4}\right)\left(\nu^2 - \frac{9}{4}\right)}{2(2x)^2} b_0 + \ldots\right] \cos x. \qquad (67)$$

This process could be continued. It is essential to note that these expressions lead to an exact result for $\nu = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$ (see Bessel functions of the half-integer index). Equation (65) is said to be a *limiting* equation for equation (64) (equation (65) is derived from (64) if we proceed to the limit in the coefficient of y when $x \to \infty$). The solution of equation (65) for large x $\left(\text{in particular for } \nu = \pm \frac{2k+1}{2}\right)$ determines reasonably well the behaviour of the solution of the original equation (65). ◆

The examples show that it is not always possible to derive the behaviour of solutions of a differential equation from that of the limiting equation.

Take the function (see [19])

$$y(x) = x^{\alpha} \sin(x^{\beta} + C), \quad \alpha > 0, \quad \beta > 0 \qquad (68)$$

and construct a differential equation for which $y(x)$ is a solution. We have

$$y' = \alpha x^{\alpha-1}\sin(x^\beta + C) + \beta x^{\alpha+\beta-1}\cos(x^\beta + C);$$

$$y'' = \left[\frac{\alpha(\alpha-1)}{x^2} - \frac{\beta^2}{x^{2-2\beta}}\right] x^\alpha \sin(x^\beta + C) + \\ + \beta(2\alpha + \beta - 1)\, x^{\alpha+\beta-2}\cos(x^\beta + C).$$

We require that α and β should be connected by the relation

$$\beta(2\alpha + \beta - 1) = 0. \tag{69}$$

The given function $y(x)$ will then satisfy the differential equation

$$y'' + \left[\frac{\beta^2}{x^{2-2\beta}} + \frac{\alpha(1-\alpha)}{x^2}\right] y = 0. \tag{70}$$

Suppose that $0 < \beta < 1$, for example $\beta = 1/2$. From condition (69) we then find that $\alpha = 1/4$, and the solution

$$y(x) = \sqrt[4]{x}\sin\left(\sqrt{x} + C\right) \tag{71}$$

of equation (70) will be oscillatory for $x \to +\infty$. On the other hand,

$$\lim_{x\to+\infty}\left[\frac{\beta^2}{x^{2-2\beta}} + \frac{\alpha(1-\alpha)}{x^2}\right] = 0,$$

therefore the limiting equation corresponding to equation (70) is

$$y'' = 0. \tag{72}$$

Its general solution

$$y = Ax + B \tag{73}$$

is without an oscillatory part.

So it is impossible to "guess" the asymptotic behaviour of solution (71) of the differential equation (70) from that of solution (73) of the limiting equation (72).

We shall cite some pertinent results. Suppose we are given the differential equation

$$y'' + p_1(x)\,y' + p_2(x)\,y = 0, \tag{74}$$

where

$$p_1(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots,$$

$$p_2(x) = b_0 + \frac{b_1}{x} + \frac{b_2}{x} + \dots, \qquad (75)$$

so that

$$\lim_{x \to \infty} p_1(x) = a_0, \quad \lim_{x \to \infty} p_2(x) = b_0. \qquad (76)$$

In this case the limiting equation has the form

$$y'' + a_0 y' + b_0 y = 0 \qquad (77)$$

and is an equation with constant coefficients. Let λ_1, λ_2 be roots (supposed for simplicity to be distinct) of the characteristic equation

$$\lambda^2 + a_0\lambda + b_0 = 0. \qquad (78)$$

The solutions of the limiting equation are the exponents $e^{\lambda_1 x}$, $e^{\lambda_2 x}$.

It turns out [13] that the asymptotic behaviour of solutions of equation (74) is similar not to the behaviour of linear combinations of the exponents $e^{\lambda_1 x}$, $e^{\lambda_2 x}$, but to that of linear combinations of the functions

$$e^{\lambda_1 x} x^{\sigma_1}, \quad e^{\lambda_2 x} x^{\sigma_2}, \qquad (79)$$

where the exponents σ_1, σ_2 are defined by the formulas

$$\sigma_1 = -\frac{a_1\lambda_1 + b_1}{a_0 + 2\lambda_1}, \quad \sigma_2 = -\frac{a_1\lambda_2 + b_1}{a_0 + 2\lambda_2}. \qquad (80)$$

Functions (79) depend not only on a_0 and b_0, i.e. not only on the limiting values of $p_1(x)$ and $p_2(x)$ for $x \to +\infty$, but also on the coefficients a_1, b_1 participating in the right-hand sides of equations (75).

Theorem. *If the characteristic equation* $\lambda^2 + a_0\lambda + b_0 = 0$ *has distinct roots* λ_1 *and* λ_2 *and if*

$$\sigma_1 = -\frac{a_1\lambda_1 + b_1}{2\lambda_1 + a_0}, \quad \sigma_2 = -\frac{a_1\lambda_2 + b_1}{2\lambda_2 + a_0},$$

then the equation

$$y'' + \left(a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots\right) y' + \left(b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \dots\right) y = 0$$

possesses linearly independent solutions $y_1(x)$ and $y_2(x)$ which can be represented by asymptotic series

$$y_1(x) \sim e^{\lambda_1 x} x^{\sigma_1}\left(1+\frac{A_1}{x}+\frac{A_2}{x^2}+\ldots\right),$$
$$y_2(x) \sim e^{\lambda_2 x} x^{\sigma_2}\left(1+\frac{B_1}{x}+\frac{B_2}{x}+\ldots\right). \quad (81)$$

If the roots of the characteristic equation coincide, then a logarithmic term may appear. It is possible to represent the solution $y_1(x)$ by an asymptotic series of the type of the first series in (81), whereas the other solution, $y_2(x)$, can now be represented by the series of the form

$$y_2(x) \sim Ay_1(x)\ln x + e^{\lambda_1 x} x^{\sigma_1}\left(K_0+\frac{K_1}{x}+\frac{K_2}{x^2}+\ldots\right). \quad (82)$$

Here the coefficients A_i, B_i, K_i can be found by the familiar method of undetermined coefficients substituting expressions (81) and (82) in the equation and equating to zero the coefficients of the powers of $1/x$, the formal differentiation of asymptotic power series, the validity of which is not a priori clear, leading to proper asymptotic representations of desired functions.

765. Show that the equation $y''+\left(1+\frac{\alpha}{x^2}\right)y=0$ has two solutions of the form

$$y_1(x)=\left(1+O\left(\frac{1}{x}\right)\right)\cos x,\ y_2(x)=\left(1+O\left(\frac{1}{x}\right)\right)\sin x$$

for $x\to+\infty$.

766. Show that when $x\to+\infty$ the equation $y''-\left(1-\frac{\alpha}{x^2}\right)y=0$ has two solutions of the form

$$y_1(x)=\left(1+O\left(\frac{1}{x}\right)\right)e^{x},\quad y_2(x)=\left(1+O\left(\frac{1}{x}\right)\right)e^{-x}.$$

CHAPTER III

SYSTEMS OF DIFFERENTIAL EQUATIONS

19. Basic concepts and definitions

The system of differential equations

$$F_k(x, y_1, y_1', \ldots, y_1^{(k_1)}, y_2, y_2', \ldots, y_2^{(k_2)}, \ldots, y_n, y_n', \ldots, y^{(k_n)}) = 0$$
$$k = 1, 2, \ldots, n, \quad (1)$$

solved for the higher derivatives $y_1^{(k_1)}, y_2^{(k_2)}, \ldots, y_n^{(k_n)}$, is called *a canonical system*. It is of the form

$$\begin{cases} y_1^{(k_1)} = f_1(x, y_1, y_1', \ldots, y_1^{(k_1-1)}, y_2, y_2', \\ \qquad \ldots, y_2^{(k_2-1)}, \ldots, y_n, y_n', \ldots, y^{(k_n-1)}), \\ y_2^{(k_2)} = f_2(x, y_1, y_1', \ldots, y_1^{(k_1-1)}), y_2, y_2', \\ \qquad \ldots, y_2^{(k_2-1)}, \ldots, y_n, y_n', \ldots, y_n^{(k_n-1)}), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n^{(k_n)} = f_n(x, y_1, y_1', \ldots, y_1^{(k_1-1)}, y_2, y_2', \\ \qquad \ldots, y_2^{(k_2-1)}, \ldots, y_n, y_n', \ldots y_n^{(k_n-1)}. \end{cases} \quad (2)$$

A number p equal to

$$p = k_1 + k_2 + \ldots + k_n$$

is called *the order of system* (1).

Example 1. Bring into the canonical form the system of equations

$$\begin{cases} y_2 y_1' - \ln(y_1'' - y_1) = 0, \\ e^{y_2'} - y_1 - y_2 = 0. \end{cases}$$

Solution. The given system is of the third order since $k_1 = 2$, $k_2 = 1$ and so $p = 3$. Solving the first equation

for y_1'' and the second equation for y_2' we obtain the canonical system

$$y_1'' = y_1 + e^{y_2 y_1'}, \quad y_2' = \ln (y_1 + y_2). \ \blacklozenge$$

A system of first order differential equations of the form

$$\frac{dx_k}{dt} = f_k (t, x_1, x_2, \ldots, x_n), \quad k = 1, 2, \ldots, n, \tag{3}$$

where t is an independent variable and $x_1, x_2, \ldots, x_n$ are unknown functions of t, is called *a normal system*.

The number n is called *the order of the normal system* (3). Two systems of differential equations are said to be *equivalent* if they have the same solutions.

Any canonical system (2) can be reduced to the equivalent normal system (3), the order of the systems remaining the same.

Example 2. Reduce to the normal system the following system of differential equations:

$$\begin{cases} \dfrac{d^2x}{dt^2} - y = 0, \\ t^3 \dfrac{dy}{dt} - 2x = 0. \end{cases}$$

Solution. We set $x = x_1$, $\frac{dx}{dt} = x_2$, $y = x_3$. Then we have $\frac{dx_1}{dt} = x_2$, $\frac{dy}{dt} = \frac{dx_3}{dt}$ and the given system reduces to the following normal system of the third order:

$$\begin{cases} \dfrac{dx_1}{dt} = x_2, \\ \dfrac{dx_2}{dt} = x_3, \\ \dfrac{dx_3}{dt} = \dfrac{2x_1}{t^3}. \end{cases}$$

Example 3. Reduce the differential equation

$$\frac{d^2x}{dt^2} + p(t) \frac{dx}{dt} + q(t)\, x = 0$$

to the normal system.

Solution. We set $x = x_1$, $\frac{dx}{dt} = x_2$, then $\frac{dx_1}{dt} = x_2$, $\frac{d^2x}{dt^2} = \frac{dx_2}{dt}$. Substituting these expressions in the given equa-

tion we get

$$\frac{dx_2}{dt}+p(t)\,x_2+q(t)\,x_1=0.$$

The normal system will have the form

$$\frac{dx_1}{dt}=x_2, \quad \frac{dx_2}{dt}=-p(t)\,x_2-q(t)\,x_1.$$

The aggregate of any n functions

$$x_1=\varphi_1(t), \quad x_2=\varphi_2(t), \ \ldots, \quad x_n=\varphi_n(t)$$

defined and continuously differentiable in an interval (a, b) is said to be *a solution of system* (3) in the interval (a, b) if they transform the equations of system (3) into identities valid for all values of $t \in (a, b)$.

Example 4. Show that the system of functions $x_1 = -1/t^2$, $x_2 = -t \ln t$ defined in the interval $0 < t < +\infty$ is a solution of the system of differential equations

$$\frac{dx_1}{dt}=2tx_1^2,$$

$$\frac{dx_2}{dt}=\frac{x_2}{t}-1.$$

Solution. We have $\frac{dx_1}{dt}=\frac{2}{t^3}$, $\frac{dx_2}{dt}=-1-\ln t$. Substituting the expressions for x_1, x_2, $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$ in terms of t in the equations of the given system we get the identities

$$\frac{2}{t^3}\equiv\frac{2t}{t^4}\equiv\frac{2}{t^3}, \quad -\ln t-1\equiv-\ln t-1, \quad 0<t<+\infty.$$

Examine if the given systems of functions are solutions of the given differential equations.

767. $\begin{cases} \dfrac{dx_1}{dt}=-2tx_1^2, \\ \dfrac{dx_2}{dt}=\dfrac{x_2+t}{t}; \end{cases} \quad \begin{cases} x_1=\dfrac{1}{t^2} \\ x_2=t\ln t. \end{cases}$

768. $\begin{cases} \dfrac{dx_1}{dt}=e^{t-x_1}, \\ \dfrac{dx_2}{dt}=2e^{x_1}; \end{cases} \quad \begin{cases} x_1=t, \\ x_2=2e^t. \end{cases}$

769. $\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = \dfrac{y^2}{x}; \end{cases}$ $\begin{cases} x = e^{2t}, \\ y = e^{t}. \end{cases}$

770. $\begin{cases} \dfrac{dy}{dx} = \dfrac{z-1}{z}, \\ \dfrac{dz}{dx} = y - x; \end{cases}$ $\begin{cases} y = x + e^{x}, \\ z = e^{-x}. \end{cases}$

The name of *the Cauchy problem* for system (3) is given to the problem of finding the solution

$$x_1 = x_1(t), \quad x_2 = x_2(t), \ \ldots, \ x_n = x_n(t)$$

of this system satisfying the initial conditions

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \ \ldots, \ x_n(t_0) = x_n^0, \tag{4}$$

$t_0, x_1^0, x_2^0, \ldots, x_n^0$ being the given numbers.

The existence and uniqueness theorem for the Cauchy problem. *Consider a normal system of differential equations* (3) *and functions* $f_i(t, x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$, *defined in some* $(n+1)$*-dimensional domain* D *of the variables* $t, x_1, x_2, \ldots, x_n$. *If there is a neighbourhood* Ω *of a point* $M_0(t_0, x_1^0, x_2^0, \ldots, x_n^0)$ *in which the functions* f_i (a) *are continuous and* (b) *have bounded partial derivatives with respect to the variables* $x_1, x_2, \ldots, x_n$, *then there is a range* $t_0 - h < t < t_0 + h$ *of* t *in which there exists a unique solution of the normal system* (3) *satisfying the initial conditions* (4).

A system of n differentiable functions

$$x_i = x_i(t, C_1, C_2, \ldots, C_n), \quad i = 1, 2, \ldots, n \tag{5}$$

of the independent variable t, and n arbitrary constants $C_1, C_2, \ldots, C_n$ is said to be *the general solution* of the normal system (3) if: (i) for any allowed values of $C_1, C_2, \ldots, C_n$ the system of functions (5) transforms equations (3) into identities, (ii) in the domain satisfying the conditions of the Cauchy theorem functions (5) give a solution to any Cauchy problem.

Example 5. Show that the system of functions

$$\begin{cases} x_1(t) = C_1 e^{-t} + C_2 e^{3t}, \\ x_2(t) = 2C_1 e^{-t} - 2C_2 e^{3t} \end{cases} \tag{6}$$

is the general solution of the system of equations

$$\begin{cases} \frac{dx_1}{dt} = x_1 - x_2, \\ \frac{dx_2}{dt} = x_2 - 4x_1. \end{cases} \tag{7}$$

Solution. In this example the domain D is

$$-\infty < t < +\infty, \quad -\infty < x_1, x_2 < +\infty. \tag{8}$$

Substituting the functions $x_1(t)$ and $x_2(t)$ of (6) in the system of equations (7) we get identities in t valid for any values of the constants C_1, C_2. Thus condition (i) determining the general solution holds.

We verify the validity of condition (ii). Note that for the system of equations (7) the conditions of the existence and uniqueness theorem for the Cauchy problem hold in the whole of the domain D determined by relations (8). Therefore any triple of numbers t_0, x_1^0, x_2^0 may be taken as the initial conditions. Relations (6) will give the system

$$\begin{cases} x_1^0 = C_1e^{-t_0} + C_2e^{3t_0}, \\ x_1^0 = 2C_1e^{-t_0} - 2C_2e^{3t_0} \end{cases}$$

from which C_1, C_2 can be found.

The determinant of the system $\Delta = -4e^{2t_0} \neq 0$; it is therefore uniquely solvable for C_1, C_2 for any x_1^0, x_2^0, and t_0. This is equivalent to stating that any Cauchy problem is solvable. So the system of functions (6) is the general solution of the system of equations (7). ◆

Solutions obtained from the general solution for particular values of the constants $C_1, C_2, \ldots, C_n$ are called *particular solutions*.

Example 6. Given the general solution (6) of system (7) find a particular solution of the system satisfying the initial conditions $x_1(0) = 0$, $x_2(0) = -4$.

Solution. The problem reduces to finding such values of the constants C_1 and C_2 that the following relations shall hold

$$0 = C_1 + C_2, \quad -4 = 2C_1 - 2C_2.$$

Solving this system we find that $C_1 = -1$, $C_2 = 1$. The desired particular solution is

$$x_1(t) = -e^{-t} + e^{3t}, \quad x_2(t) = -2e^{-t} - 2e^{3t}.$$

Remark 1. Not every system of differential equations can be reduced to one equation. For example, the system

$$\begin{cases} \dfrac{dx_1}{dt} = -x_1, \\ \dfrac{dx_2}{dt} = x_2 \end{cases}$$

splits into two independent equations. In this case the general solution is obtained by integrating each equation separately:

$$x_1 = C_1 e^{-t}, \quad x_2 = C_2 e^{t}.$$

Remark 2. If the number of equations in a system is equal to n and the number of desired functions is equal to N, with $N > n$, then this system is *indeterminate*. In this case it is possible to choose arbitrarily $N - n$ desired functions (provided they can be differentiated a necessary number of times) and from them to find the remaining n functions.

Remark 3. If a system consists of n equations and the number of desired functions is N, with $N < n$, then this system may be found to be *incompatible*, i.e. to have no solution. ◆

Consider (for simplicity) a normal system of two differential equations

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(t, x_1, x_2), \\ \dfrac{dx_2}{dt} = f_2(t, x_1, x_2). \end{cases} \tag{9}$$

Let the system of values of t, x_1, x_2 be rectangular coordinates of a point in three-dimensional space referred to the coordinate system Otx_1x_2. The solution

$$x_1 = x_1(t), \quad x_2 = x_2(t),$$

taking for $t = t_0$ the values x_1^0, x_2^0, represents in that space a certain line passing through a point $M_0(t_0, x_1^0, x_2^0)$. This line is called *an integral curve* (or *line*) of the normal system (9).

Geometrically the Cauchy problem for system (9) can be stated as follows: find in the space of the variables (t, x_1, x_2) an integral curve passing through a given point (t_0, x_1^0, x_2^0).

The Cauchy theorem establishes the existence and uniqueness of such a curve.

The normal system (9) and its solution can be given the following interpretation as well. Let the independent variable t be regarded as time and the system of values of x_1, x_2 as rectangular coordinates of a point in the plane x_1Ox_2. This plane of variables, x_1Ox_2, is called *a phase plane*. In the phase plane the solution

$$x_1 = x_1(t), \quad x_2 = x_2(t)$$

of system (9) taking the initial values x_1^0, x_2^0 for $t = t_0$ is represented by the curve AB (Fig. 28) passing through the

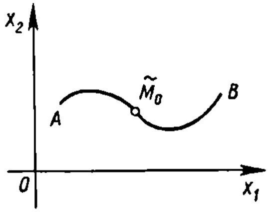

Fig. 28

point $M_0(x_1^0, x_2^0)$. This curve is called *a trajectory* of the system (*a phase trajectory*). It is obvious that the trajectory of system (9) is the projection of an integral curve onto the phase plane.

System (9) determines at a point (x_1, x_2) of the phase plane at a time t the coordinates of velocity $\{f_1, f_2\}$ of a moving point.

Example 7. Solve the system of equations

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x \tag{10}$$

for the initial conditions

$$x(0) = x_0, \quad y(0) = y_0. \tag{11}$$

Solution. Differentiating the first equation of system (10) once with respect to t and substituting $\frac{dy}{dt} = -x$ in the equation obtained we reduce system (10) to one second order equation $\frac{d^2x}{dt^2} + x = 0$ the general solution of which is

$$x = C_1 \cos t + C_2 \sin t.$$

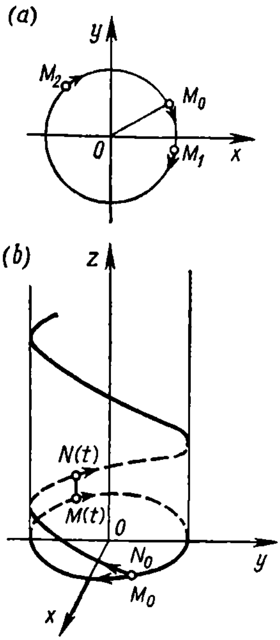

Fig. 29

Since $y = \frac{dx}{dt}$, we have $y = -C_1 \sin t + C_2 \cos t$; thus tho general solution of system (10) is

$$x = C_1 \cos t + C_2 \sin t, \quad y = -C_1 \sin t + C_2 \cos t. \tag{12}$$

The particular solution of system (10) satisfying the initial conditions (11) is

$$x = x_0 \cos t + y_0 \sin t, \quad y = -x_0 \sin t + y_0 \cos t. \tag{13}$$

Eliminating t from equations (13) (by squaring and term-by-term addition) we obtain the phase trajectory

$$x^2 + y^2 = R^2, \tag{14}$$

where $R = \sqrt{x_0^2 + y_0^2}$. This is a circle passing through the point $M_0(x_0, y_0)$. Rewriting equations (13) in the form

$$x = R \sin (t + \alpha), \quad y = R \cos (t + \alpha), \tag{15}$$

where $R = \sqrt{x_0^2 + y_0^2}$, $\sin \alpha = x_0/R$, $\cos \alpha = y_0/R$, we notice that equations (15) express the time dependence of the moving coordinates of a point $M(x(t), y(t))$, or $M(t)$ for short, which starts from the point $M_0(x_0, y_0)$ as $t = 0$ and moves round circle (14) (Fig. 29a).

We determine the direction of motion of the point $M(t)$ using the given system (10). By the equation $\frac{dy}{dt} = -x$, the variable y decreases when $x > 0$ (as, for example, at the point $M_1(t)$) and increases when $x < 0$ (as, for example, at the point $M_2(t)$). Thus the point $M(t)$ moves clockwise in curve (14). Changing arbitrarily the initial conditions (11) (keeping, however, within physically permissible limits), i.e. changing arbitrarily the position of the starting point $M_0(x_0, y_0)$ we get all kinds of phase trajectories (14).

We shall now give another interpretation to equations (15) (or, which is the same, to equations (13)). We take a right-handed Cartesian coordinate system $Oxyz$ in three-dimensional space. It is easily seen that the point $N(x(t), y(t), z(t))$ (or $N(t)$ for short) with the coordinates (Fig. 29b)

$$x(t) = R\sin(t + \alpha), \quad y(t) = R\cos(t + \alpha), \quad z(t) = t \tag{16}$$

starts at $t = 0$ from the point $N_0(x_0, y_0, 0)$ and, as t increases, it ascends helix (16) located on cylinder (14) with elements parallel to the Oz axis.

It is obvious that the point N_0 coincides with the point M_0 and that for any t the point $N(t)$ is projected into the point $M(t)$ on the phase trajectory. Since the point $M(t)$ moves clockwise, the integral curve described by the point $N(t)$ is a left-handed helix on cylinder (14). For different positions of the point $N_0(x_0, y_0, 0)$ integral curves of system (10) corresponding to different values of $R = \sqrt{x_0^2 + y_0^2}$ are projected into different curves (14) on the xOy plane and integral curves corresponding to the same value of R are projected into the same curve (14). ◆

A function $\psi(t, x_1, x_2, \ldots, x_n)$ defined and continuous together with its first order partial derivative $\frac{\partial\psi}{\partial t}, \frac{\partial\psi}{\partial x_k}$, $k = 1, 2, \ldots, n$, in a domain D is said to be *the integral of a normal system* (3) if, when an arbitrary solution $x_1(t), x_2(t), \ldots, x_n(t)$ of system (3) is substituted in it, the function takes a constant value, i.e. the function

$\psi(t, x_1, x_2, \ldots, x_n)$ depends only on the choice of a solution $x_1(t), x_2(t), \ldots, x_n(t)$, not on the variable t.

A first integral of a normal system (3) is the equation

$$\psi(t, x_1, x_2, \ldots, x_n) = C,$$

where $\psi(t, x_1, x_2, \ldots, x_n)$ is an integral of system (3) and C is an arbitrary constant*.

Example 8. Show that a function

$$\psi(t, x_1, x_2) = \frac{x_2}{t} - x_1, \tag{17}$$

defined in the domain D: $t \neq 0$, $-\infty < x_1, x_2 < +\infty$, is an integral of the system of equations

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{x_1}{t} \\ \frac{dx_2}{dt} &= x_1 + \frac{x_2}{t} \end{aligned} \tag{18}$$

if the general solution of the system is

$$x_1 = C_1 t, \quad x_2 = C_1 t^2 + C_2 t. \tag{19}$$

Solution. Substituting (19) in (17) we get

$$\psi(t, x_1, x_2) = \psi(t, C_1 t, C_1 t^2 + C_2 t) = \frac{C_1 t^2 + C_2 t}{t} - C_1 t = C_2$$

in the domain D. Hence function (17) is an integral of the system of equations (18) in the domain D and so the first integral of this system is $\frac{x^2}{t} - x_1 = C$, where C is an arbitrary constant.

Theorem. *For a function $\psi(t, x_1, x_2, \ldots, x_n)$ to be an integral of system* (3) *it is necessary and sufficient that the conditions*

$$\frac{\partial \psi}{\partial t} + \sum_{k=1}^{n} f_k(t, x_1, x_2, \ldots, x_n) \frac{\partial \psi}{\partial x_k} = 0 \tag{20}$$

should hold in the domain D.

Example 9. Show that the function

$$\psi(t, x_1, x_2) = \arctan \frac{x_1}{x_2} - t \tag{21}$$

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* Sometimes the name first integral of system (3) is given to an integral of this system.

is an integral of the system of equations

$$\frac{dx_1}{dt}=\frac{x_1^2}{x_2}, \quad \frac{dx_2}{dt}=-\frac{x_2^2}{x_1}. \tag{22}$$

Solution. In this case

$$f_1(t, x_1, x_2)=\frac{x_1^2}{x_2}, \quad f_2(t, x_1, x_2)=-\frac{x_2^2}{x_1}. \tag{23}$$

We find the partial derivatives of the given function $\psi(t, x_1, x_2)$. We have

$$\frac{\partial\psi}{\partial t}=-1, \quad \frac{\partial\psi}{\partial x_1}=\frac{x_2}{x_1^2+x_2^2}, \quad \frac{\partial\psi}{\partial x_2}=-\frac{x_1}{x_1^2+x_2^2}. \tag{24}$$

Substituting (23) and (24) in the left-hand side of (20) we get

$$\frac{\partial\psi}{\partial t}+f_1(t, x_1, x_2)\frac{\partial\psi}{\partial x_1}+f_2(t, x_1, x_2)\frac{\partial\psi}{\partial x_2}$$
$$=-1+\frac{x_1^2}{x_2}\cdot\frac{x_2}{x_1^2+x_2^2}+\frac{x_2^2}{x_1}\cdot\frac{x_1}{x_1^2+x_2^2}=-1+1=0$$

in the domain D: $-\infty<t<+\infty$, $x_1\neq 0$, $x_2\neq 0$.

Thus function (21) is an integral of the system of equations (22) and so the first integral of system (22) is

$$\arctan\frac{x_1}{x_2}-t=C,$$

where C is an arbitrary constant. ◆

The normal system (3) has an infinite number of systems of first integrals.

Integrals $\psi_1, \psi_2, \ldots, \psi_n$ of system (3) are said to be *independent with respect to the desired functions* $x_1, x_2, \ldots, x_n$ if there exists no relation of the form $F(\psi_1, \psi_2, \ldots, \psi_n)=0$ between the functions $\psi_1, \psi_2, \ldots, \psi_n$ for any choice of the function F not depending explicitly on $x_1, x_2, \ldots, x_n$.

Theorem. *For functions $\psi_1, \psi_2, \ldots, \psi_n$ having partial derivatives $\frac{\partial\psi_i}{\partial x_k}$, $i, k=1, 2, \ldots, n$, to be independent with respect to $x_1, x_2, \ldots, x_n$ in some domain D it is necessary and sufficient that the Jacobian of these functions should be*

nonzero in the domain D

$$\frac{D(\psi_1, \psi_2, \ldots, \psi_n)}{D(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial \psi_1}{\partial x_1} & \frac{\partial \psi_1}{\partial x_2} & \cdots & \frac{\partial \psi_1}{\partial x_n} \\ \frac{\partial \psi_2}{\partial x_1} & \frac{\partial \psi_2}{\partial x_2} & \cdots & \frac{\partial \psi_2}{\partial x_n} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial \psi_n}{\partial x_1} & \frac{\partial \psi_n}{\partial x_2} & \cdots & \frac{\partial \psi_n}{\partial x_n} \end{vmatrix} \neq 0.$$

The general integral of a normal system (3) is an aggregate of n independent first integrals of that system.

If k independent first integrals of system (3) are known, with $k < n$, then its order can be depressed by k units.

Examine if the given functions ψ are first integrals of the given systems of differential equations.

771. $\begin{cases} \frac{dx_1}{dt} = \frac{x_1^2}{x_2}, \\ \frac{dx_2}{dt} = x_2 - x_1; \end{cases}$ $\quad \psi = x_1 x_2 e^{-t}$.

772. $\begin{cases} \frac{dx}{dt} = \frac{e^{-x}}{t}, \\ \frac{dy}{dt} = \frac{x}{t} e^{-y}; \end{cases}$ $\quad \psi = (1+x) e^{-x} - e^{-y}$.

773. $\begin{cases} \frac{dx}{dt} = \frac{y+t}{x+y}, \\ \frac{dy}{dt} = \frac{x-t}{x+y}; \end{cases}$ $\quad$ (a) $\psi_1 = x + y - t$, (b) $\psi_2 = x + y + t$.

Examine for the systems of differential equations below if the given pairs of functions form systems of independent first integrals:

774. $\begin{cases} \frac{dx}{dt} = \frac{t-y}{y-x}, \\ \frac{dy}{dt} = \frac{x-t}{y-x}; \end{cases}$ $\quad x + y + t = C_1$, $\quad x^2 + y^2 + t^2 = C_2$.

775. $\begin{cases} \frac{dx}{dt} = \frac{t+y}{x+y}, \\ \frac{dy}{dt} = \frac{t+x}{x+y}; \end{cases}$ $\quad \frac{x-y}{t-x} = C_1$, $\quad \frac{t-x}{t-y} = C_2$.

20. The method of elimination (reducing a system of differential equations to a single equation)

A particular case of a canonical system of differential equations is one equation of the nth order solved for a higher derivative

$$x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}).$$

Introducing new functions

$$x_1 = x'(t), \quad x_2 = x''(t), \ldots, x_{n-1} = x^{(n-1)}(t)$$

replaces this equation by a normal system of n equations

$$\begin{cases} \dfrac{dx}{dt} = x_1, \\ \dfrac{dx_1}{dt} = x_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{dx_{n-2}}{dt} = x_{n-1}, \\ \dfrac{dx_{n-1}}{dt} = f(t, x, x_1, x_2, \ldots, x_{n-1}). \end{cases}$$

The inverse statement is true, that in general a normal system of n first order equations

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(t, x_1, x_2, \ldots, x_n) \\ \dfrac{dx_2}{dt} = f_2(t, x_1, x_2, \ldots, x_n), \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{dx_n}{dt} = f_n(t, x_1, x_2, \ldots, x_n) \end{cases}$$

is equivalent to one equation of order n. This provides a basis for one of the methods of integrating systems of differential equations, for *the method of elimination.*

We shall illustrate this method by considering a system of two equations:

$$\frac{dx}{dt} = ax + by + f(t), \qquad \frac{dy}{dt} = cx + dy + g(t). \tag{1}$$

Here a, b, c, d are constants, $f(t)$ and $g(t)$ are given functions, and $x(t)$ and $y(t)$ are the desired functions. From

the first equation of system (1) we find that

$$y=\frac{1}{b}\left(\frac{dx}{dt}-ax-f(t)\right). \tag{2}$$

Substituting in the second equation the right-hand side of (2) for y and the derivative of the right hand-side of (2) for $\frac{dy}{dt}$ we obtain an equation of the second order in $x(t)$

$$A\frac{d^2x}{dt^2}+B\frac{dx}{dt}+Cx+P(t)=0,$$

A, B, C being constants. Hence we find that $x=x(t, C_1, C_2)$. By substituting the expression obtained for x and $\frac{dx}{dt}$ in (2) we find y.

Example 1. Integrate the system of equations

$$\begin{cases} \dfrac{dx}{dt}=y+1, \\ \dfrac{dy}{dt}=x+1. \end{cases} \tag{3}$$

Solution. From the first equation of system (3) we find that $y=\frac{dx}{dt}-1$, then

$$\frac{dy}{dt}=\frac{d^2x}{dt^2}. \tag{4}$$

Substituting (4) in the second equation of system (3) we obtain a linear differential equation of the second order with constant coefficients

$$\frac{d^2x}{dt^2}-x-1=0. \tag{5}$$

The general solution of equation (5) is

$$x=C_1e^t+C_2e^{-t}-1. \tag{6}$$

Finding the derivative of (6) with respect to t we get

$$y=\frac{dx}{dt}-1=C_1e^t+C_2e^{-t}-1.$$

The general solution of system (3) is

$$x=C_1e^t+C_2e^{-t}-1, \quad y=C_1e^t-C_2e^{-t}-1.$$

Example 2. Solve the Cauchy problem for the system

$$\begin{cases} \dfrac{dx}{dt} = 3x + 8y, \\ \dfrac{dy}{dt} = -x - 3y, \end{cases} \tag{7}$$

$$x(0) = 6, \quad y(0) = -2. \tag{8}$$

Solution. From the second equation of system (7) we find that

$$x = -3y - \frac{dy}{dt}, \tag{9}$$

whence

$$\frac{dx}{dt} = -3\frac{dy}{dt} - \frac{d^2y}{dt^2}. \tag{10}$$

Substituting (9) and (10) in the first equation of system (7) we obtain the equation $\frac{d^2y}{dt^2} - y = 0$ whose general solution is

$$y = C_1 e^t - C_2 e^{-t}. \tag{11}$$

Substituting (11) in (9) we find that

$$x = -4C_1 e^t - 2C_2 e^{-t}.$$

The general solution of system (7) is

$$x = -4C_1 e^t - 2C_2 e^{-t}, \quad y = C_1 e^t + C_2 e^{-t}. \tag{12}$$

For the initial conditions (8) we obtain from (12) a system of equations to determine C_1, C_2:

$$\begin{cases} 6 = -4C_1 - 2C_2, \\ -2 = C_1 + C_2, \end{cases}$$

solving which we find that $C_1 = -1$, $C_2 = -1$. Substituting the values of C_1 and C_2 in (12) we obtain the solution of the Cauchy problem set:

$$x = 4e^t + 2e^{-t}, \quad y = -e^t - e^{-t}.$$

Example 3. Solve the system of equations

$$\begin{cases} t\dfrac{dx}{dt} = -x + yt, \\ t^2\dfrac{dy}{dt} = -2x + yt. \end{cases}$$

Solution. From the first equation of the system we find that

$$y=\frac{x}{t}+\frac{dx}{dt}$$

so that

$$\frac{dy}{dt}=-\frac{x}{t^2}+\frac{1}{t}\frac{dx}{dt}+\frac{d^2x}{dt^2}.$$

Substituting these expressions for y and $\frac{dy}{dt}$ in the second equation we get

$$t^2\frac{d^2x}{dt^2}+t\frac{dx}{dt}-x=-2x+x+t\frac{dx}{dt} \quad \text{or} \quad t^2\frac{d^2x}{dt^2}=0.$$

Assuming $t\neq 0$, we get $\frac{d^2x}{dt^2}=0$ from the last equation and on integrating we find that $x=C_1+C_2t$. Now we readily get

$$y=\frac{x}{t}+\frac{dx}{dt}=\frac{C_1+C_2t}{t}+C_2=2C_2+\frac{C_1}{t}.$$

The general solution of the given system is

$$x=C_1+C_2t, \quad y=\frac{C_1}{t}+2C_2, \quad t\neq 0.$$

Solve the following systems of differential equations by the method of elimination:

776. $\begin{cases} \dfrac{dx}{dt}=-9y, \\ \dfrac{dy}{dt}=x, \end{cases}$

777. $\begin{cases} \dfrac{dx}{dt}=y+t, \\ \dfrac{dy}{dt}=x-t. \end{cases}$

778. $\begin{cases} \dfrac{dx}{dt}+3x+4y=0, \\ \dfrac{dy}{dt}+2x+5y=0, \quad x(0)=1, \quad y(0)=4. \end{cases}$

779. $\begin{cases} \dfrac{dx}{dt}=x+5y, \\ \dfrac{dy}{dt}=-x-3y, \quad x(0)=-2, \quad y(0)=1. \end{cases}$

780. $\begin{cases} 4\dfrac{dx}{dt} - \dfrac{dy}{dt} + 3x = \sin t, \\ \dfrac{dx}{dt} + y = \cos t. \end{cases}$

781. $\begin{cases} \dfrac{dx}{dt} = -y + z, \\ \dfrac{dy}{dt} = z, \\ \dfrac{dz}{dt} = -x + z. \end{cases}$

782. $\begin{cases} \dfrac{dx}{dt} = y + z, \\ \dfrac{dy}{dt} = x + z, \\ \dfrac{dz}{dt} = x + y. \end{cases}$

783. $\begin{cases} \dfrac{d^2x}{dt^2} = y, \\ \dfrac{d^2y}{dt^2} = x. \end{cases}$

784. $\begin{cases} \dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} + x = 0, \\ \dfrac{dx}{dt} + \dfrac{d^2y}{dt^2} = 0. \end{cases}$

785. $\begin{cases} \dfrac{d^2x}{dt^2} = 3x + y, \\ \dfrac{dy}{dt} = -2x. \end{cases}$

786. $\begin{cases} \dfrac{d^2x}{dt^2} = x^2 + y, \\ \dfrac{dy}{dt} = -2x\dfrac{dx}{dt} + x, \quad x(0) = x'(0) = 1, \quad y(0) = 0. \end{cases}$

21. Finding integrable combinations. A symmetrical form of a system of differential equations

21.1. Finding integrable combinations. This method of integrating a system of differential equations

$$\frac{dx_k}{dt} = f_k(t, x_1, x_2, \ldots, x_n), \quad k = 1, 2, \ldots, n \tag{1}$$

is as follows: using suitable arithmetic operations (addition, subtraction, multiplication, and division) one forms from system (1) the so-called integrable combinations, i.e. equations of the form

$$F\left(t,\ u,\ \frac{du}{dt}\right)=0,$$

that can be solved easily enough, u being some function of the desired function $x_1(t)$, $x_2(t)$, ..., $x_n(t)$. Each integrable combination gives one first integral. If n independent first integrals of system (1) are obtained, then its integration is complete; if, however, m independent first integrals are obtained, where $m<n$, then system (1) is reduced to a system with a smaller number of unknown functions.

Example 1. Solve the system

$$\begin{cases} \dfrac{dx_1}{dt}=2(x_1^2+x_2^2)t, \\ \dfrac{dx_2}{dt}=4x_1x_2t. \end{cases} \tag{2}$$

Solution. Adding termwise both equations we get

$$\frac{d(x_1+x_2)}{dt}=(x_1+x_2)^2\,2t,$$

whence

$$-\frac{1}{x_1+x_2}=t^2-C_1 \text{ or } \frac{1}{x_1+x_2}+t^2=C_1.$$

Subtracting termwise the second equation from the first we get

$$\frac{d(x_1-x_2)}{dt}=2t(x_1-x_2)^2,$$

whence

$$\frac{1}{x_1-x_2}+t^2=C_1.$$

Thus we have found two first integrals of the given system

$$\psi_1(t,\ x_1,\ x_2)=t^2+\frac{1}{x_1+x_2}=C_1,$$

$$\psi_2(t,\ x_1,\ x_2)=t^2+\frac{1}{x_1-x_2}=C_2,$$

which are independent, since the Jacobian

$$\frac{D(\psi_1, \psi_2)}{D(x_1, x_2)} = \begin{vmatrix} \frac{\partial \psi_1}{\partial x_1} & \frac{\partial \psi_1}{\partial x_2} \\ \frac{\partial \psi_2}{\partial x_1} & \frac{\partial \psi_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} -\frac{1}{(x_1+x_2)^2} & -\frac{1}{(x_1+x_2)^2} \\ -\frac{1}{(x_1-x_2)^2} & -\frac{1}{(x_1-x_2)^2} \end{vmatrix}$$

$$= -\frac{2}{(x_1^2-x_2^2)^2} \neq 0.$$

The general integral of system (2) is

$$t^2 + \frac{1}{x_1+x_2} = C_1, \quad t^2 + \frac{1}{x_1-x_2} = C_2. \tag{3}$$

Solving system (3) for the unknown functions we obtain the general solution of system (2):

$$x_1 = \frac{C_1+C_2-2t^2}{2(C_1-t^2)(C_2-t^2)}, \quad x_2 = \frac{C_2-C_1}{2(C_1-t^2)(C_2-t^2)}.$$

Example 2. Solve the system

$$\begin{cases} \frac{dx_1}{dt} = \frac{x_1-x_2}{x_3-t}, \\ \frac{dx_2}{dt} = \frac{x_1-x_2}{x_3-t}, \\ \frac{dx_3}{dt} = x_1-x_2+1. \end{cases} \tag{4}$$

Solution. Subtracting the second equation from the first we get $\frac{d(x_1-x_2)}{dt} = 0$, whence the first integral of system (4) is

$$x_1 - x_2 = C_1. \tag{5}$$

Substituting (5) in the second and third equations of system (4) we obtain a system with two unknown functions

$$\begin{cases} \frac{dx_2}{dt} = \frac{C_1}{x_3-t}, \\ \frac{dx_3}{dt} = C_1+1. \end{cases} \tag{6}$$

From the second equation of system (6) we find that

$$x_3 = (C_1+1)\,t + C_2. \tag{7}$$

Substituting (7) in the first equation of system (6) we have

$$\frac{dx_2}{dt} = \frac{C_1}{C_1 t + C_2}, \quad x_2 = \ln|C_1 t + C_2| + C_3;$$

thus

$$x_1 - x_2 = C_1, \quad x_2 = \ln | C_1 t + C_2 | + C_3, \quad x_3 = (C_1 + 1) t + C_2.$$

From this we find the general solution of system (4):

$$x_1 = \ln | C_1 t + C_2 | + C_1 + C_3, \quad x_2 = \ln | C_1 t + C_2 | + C_3, \quad x_3 = (C_1 + 1) t + C_2.$$

Example 3. Find a particular solution of the system

$$\begin{cases} \dfrac{dx}{dt} = 1 - \dfrac{1}{y}, \\ \dfrac{dy}{dt} = \dfrac{1}{x-t}. \end{cases}$$

satisfying the initial conditions $x|_{t=0} = 1$, $y|_{t=0} = 1$.

Solution. We write the given system in the form

$$\begin{cases} y\left(\dfrac{dx}{dt} - 1\right) = -1, \\ (x-t)\dfrac{dy}{dt} = 1, \end{cases} \quad \text{or} \quad \begin{cases} y\dfrac{d(x-t)}{dt} = -1, \\ (x-t)\dfrac{dy}{dt} = 1. \end{cases}$$

Adding termwise the last equations we get

$$y\frac{d(x-t)}{dt} + (x-t)\frac{dy}{dt} = 0 \quad \text{or} \quad \frac{d}{dt}[(x-t)y] = 0.$$

Hence we have the first integral $(x-t)y = C_1$. Since $x - t = C_1/y$, the second equation of the system becomes $\frac{dy}{dt} = \frac{y}{C_1}$, whence $y = C_2 e^{t/C_1}$. Thus

$$(x - t)y = C_1, \quad y = C_2 e^{t/C_1},$$

whence we obtain the general solution

$$x = t + \frac{C_1}{C_2} e^{-t/C_1}, \quad y = C_2 e^{t/C_1}.$$

Setting $t = 0$ in these last equations we get $1 = C_1/C_2$, $1 = C_2$, i.e. $C_1 = C_2 = 1$, and the desired particular solution is

$$x = t + e^{-t}, \quad y = e^t.$$

Example 4 (decomposition of matter). A substance A decomposes into two substances X and Y at a formation rate

proportional to the amount of undecomposed substance. Find the law of change for the amounts x and y of the substances X and Y as function of time t, if we have $x = y = 0$ at $t = 0$ and $x = a/8$, $y = 3a/8$ at the end of an hour, a being the initial amount of the substance A.

Solution. At time t the amount of undecomposed substance A is $a - x - y$. From the statement of the problem we have

$$\begin{cases} \frac{dx}{dt} = k_1 (a - x - y), \\ \frac{dy}{dt} = k_2 (a - x - y). \end{cases} \tag{8}$$

Dividing termwise the second equation by the first we get

$$\frac{dy}{dx} = \frac{k_2}{k_1}, \quad \text{whence} \quad y = \frac{k_2}{k_1} x + C_1.$$

For $t = 0$ we have $x = y = 0$, therefore we find from the last equation that $C_1 = 0$ and so

$$y = \frac{k_2}{k_1} x. \tag{9}$$

On substituting (9) in the first equation of the system we obtain the equation

$$\frac{dx}{dt} + (k_1 + k_2) x = k_1 a$$

the general solution of which is

$$x = \frac{k_1 a}{k_1 + k_2} + C_2 e^{-(k_1 + k_2)t}.$$

Using the initial condition $x|_{t=0} = 0$ we find that $C_2 = -\frac{k_1 a}{k_1 + k_2}$ so that

$$x = \frac{k_1 a}{k_1 + k_2} [1 - e^{-(k_1 + k_2)t}]. \tag{10}$$

Substituting (10) in (9) we have

$$y = \frac{k_2 a}{k_1 + k_2} [1 - e^{-(k_1 + k_2)t}].$$

To determine the coefficients k_1 and k_2 we take an hour as a unit of time. Recalling that $x = \frac{a}{8}$, $y = \frac{3}{8} a$ for $t = 1$

we find that

$$\frac{k_1}{k_1+k_2}[1-e^{-(k_1+k_2)}]=\frac{1}{8}, \quad \frac{k_2}{k_1+k_2}[1-e^{-(k_1+k_2)}]=\frac{3}{8},$$

whence

$$k_2=3k_1, \quad k_1+k_2=\ln 2$$

so that $k_1=\frac{\ln 2}{4}$, $k_2=\frac{3}{4}\ln 2$ and the desired solution of system (8) is

$$x=\frac{a}{4}(1-2^{-t}), \quad y=\frac{3a}{4}(1-2^{-t}).$$

Example 5 (equilibrium of gases in communicating vessels). Let there be two vessels, with volumes V_1 and V_2 respectively, filled with gas. The gas pressure at the initial time is P_1 in the first vessel and P_2 in the second. The vessels are connected by a pipe through which the gas flows from one vessel into the other. Considering that the rate of gas flow through the pipe is proportional to the difference of the squares of the pressures, determine the pressures p_1 and p_2 in the vessels at time t.

Solution. Let a be the rate of flow of the gas for the difference of pressures equal to unity. Then the amount of the gas that will flow from one vessel into the other during the time dt is $a(p_1^2-p_2^2)\,dt$. This amount is equal to the decrease of gas during the time dt in one vessel and to the increase of gas during the same time in the other. This can be expressed by the system of equations

$$\left\{\begin{aligned} a(p_1^2-p_2^2)&=bV_2\frac{dp_2}{dt},\\ a(p_1^2-p_2^2)&=-bV_1\frac{dp_1}{dt},\end{aligned}\right. \tag{11}$$

b being a constant coefficient.

Subtracting the second equation of system (11) from the first we get

$$V_1\frac{dp_1}{dt}+V_2\frac{dp_2}{dt}=0,$$

whence

$$V_1p_1+V_2p_2=C_1. \tag{12}$$

We multiply both sides of the first equation of system (11) by p_1V_1 and those of the second equation by p_2V_2 and add termwise:

$$a(p_1^2-p_2^2)(p_1V_1+p_2V_2)=bV_1V_2\left(p_1\frac{dp_2}{dt}-p_2\frac{dp_1}{dt}\right). \qquad (13)$$

Taking into account (12) and dividing both sides of (13) by p_1^2 we have

$$\frac{d}{dt}\left(\frac{p_2}{p_1}\right)=\left[1-\left(\frac{p_2}{p_1}\right)^2\right]k,$$

where $k=\frac{aC_1}{bV_1V_2}$. Denoting $p_2/p_1=z$ we get

$$\frac{dz}{1-z^2}=k\,dt, \quad \text{whence} \quad \ln\left|\frac{1+z}{1-z}\right|=2kt+\ln C_2$$

or

$$\frac{1+z}{1-z}=C_2e^{2kt}. \qquad (14)$$

Substituting the quantity p_2/p_1 for z in (14) we finally get

$$\frac{p_1+p_2}{p_1-p_2}=C_2e^{2kt}. \qquad (15)$$

At the initial time $t=0$ we have $p_1=P_1$, $p_2=P_2$, so that from equation (12) we have

$$C_1=P_1V_1+P_2V_2, \qquad (16)$$

and from equation (15) we have

$$C_2=\frac{P_1+P_2}{P_1-P_2}. \qquad (17)$$

Equations (12) and (15) give the desired pressures $p_1(t)$ and $p_2(t)$ at any time t, the constants C_1 and C_2 being determined by formulas (16) and (17).

Solve the following systems of differential equations:

787. $\begin{cases} \frac{dx}{dt}=x^2+y^2, \\ \frac{dy}{dt}=2xy. \end{cases}$

788. $\begin{cases} \frac{dx}{dt}=-\frac{1}{y}, \\ \frac{dy}{dt}=\frac{1}{x}. \end{cases}$

789. $\begin{cases} \frac{dx}{dt}=\frac{x}{y}, \\ \frac{dy}{dt}=\frac{y}{x}. \end{cases}$

790. $\begin{cases} \frac{dx}{dt}=\frac{y}{x-y}, \\ \frac{dy}{dt}=\frac{x}{x-y}. \end{cases}$

791. $\begin{cases} \dfrac{dx}{dt} = \sin x \cos y, \\ \dfrac{dy}{dt} = \cos x \sin y. \end{cases}$ **792.** $\begin{cases} e^t \dfrac{dx}{dt} = \dfrac{1}{y}, \\ e^t \dfrac{dy}{dt} = \dfrac{1}{x}. \end{cases}$

793. $\begin{cases} \dfrac{dx}{dt} = \cos^2 x \cos^2 y + \sin^2 x \cos^2 y, \\ \dfrac{dy}{dt} = -\dfrac{1}{2} \sin 2x \sin 2y, \quad x(0) = 0, \quad y(0) = 0. \end{cases}$

21.2. A symmetrical form of a system of differential equations. To find integrable combinations in solving the system of differential equations (1) it is sometimes convenient to write it in the symmetrical form

$$\frac{dx_1}{f_1(t, x_1, x_2, \ldots, x_n)} = \frac{dx_2}{f_2(t, x_1, x_2, \ldots, x_n)} = \ldots = \frac{dx_n}{f_n(t, x_1, x_2, \ldots, x_n)} = \frac{dt}{1}. \tag{18}$$

In the system of differential equations written in the symmetrical form the variables $t, x_1, x_2, \ldots, x_n$ are equivalent, which in some cases simplifies finding integrable combinations.

To solve system (18) one may either take pairs of relations allowing separation of the variables or use the derived proportions

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots + \frac{a_m}{b_m} = \frac{\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_m a_m}{\lambda_1 b_1 + \lambda_2 a_2 + \ldots + \lambda_m a_m}, \tag{19}$$

where the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_m$ are arbitrary and chosen so that the numerator should be the differential of the denominator or so that the numerator should be the total differential and the denominator should be equal to zero.

Example 6. Find the general solution of the system of equations

$$\frac{dt}{2x} = \frac{dx}{-\ln t} = \frac{dy}{\ln t - 2x}. \tag{20}$$

Solution. The first integrable combination is $\dfrac{dt}{2x} = -\dfrac{dx}{\ln t}$. Separating the variables and integrating we find the first integral

$$t(\ln t - 1) + x^2 = C_1. \tag{21}$$

The second integrable combination can be obtained using the derived proportions (19). To do this we add the numerators and denominators of fractions in system (20):

$$\frac{dt}{2x}=\frac{dx}{-\ln t}=\frac{dy}{\ln t-2x}=\frac{dt+dx+dy}{0},$$

here $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$. Hence $dt + dx + dy = 0$ or $d(t + x + y) = 0$ and so

$$t + x + y = C_2. \tag{22}$$

The first integrals (21) and (22) give the general integral of system (20)

$$x^2 + t(\ln t - 1) = C_1, \quad x + y + t = C_2,$$

from which we find the general solution of the system

$$x=\pm\sqrt{C_1+t(\ln t-1)}, \quad y=C_2-t\pm\sqrt{C_1+t(\ln t-1)}.$$

Example 7. Solve the system of equations

$$\frac{dt}{4y-5x}=\frac{dx}{5t-3y}=\frac{dy}{3x-4t}. \tag{23}$$

Solution. Multiplying the numerators and denominators in system (23) by 3, 4, 5 respectively and adding the numerators and denominators we get by (19)

$$\frac{3\,dt}{12y-15x}=\frac{4\,dx}{20t-12y}=\frac{5\,dy}{15x-20t}=\frac{3\,dt+4\,dx+5\,dy}{0}$$

(here $\lambda_1 = 3$, $\lambda_2 = 4$, $\lambda_3 = 5$). Hence $3dt + 4dx + 5dy = 0$ or $d(3t + 4x + 5y) = 0$ and so $3t + 4x + 5y = C_1$ is a first integral of system (23).

Multiplying the numerators and denominators of fractions in system (23) by $\lambda_1 = 2t$, $\lambda_2 = 2x$, $\lambda_3 = 2y$ respectively and adding the numerators and denominators we obtain by (19)

$$\frac{2t\,dt}{8yt-10xt}=\frac{2x\,dx}{10tx-6yx}=\frac{2y\,dy}{6xy-8ty}=\frac{2t\,dt+2x\,dx+2y\,dy}{0};$$

hence

$$2t\,dt + 2x\,dx + 2y\,dy = 0 \quad \text{or} \quad d(t^2 + x^2 + y^2) = 0$$

and so the second first integral is $t^2 + x^2 + y^2 = C_2$.

The aggregate of the first integrals that are independent gives the general integral of system (23):

$$3t + 4x + 5y = C_1, \quad t^2 + x^2 + y^2 = C_2.$$

Thus system (23) is solved.

Solve the following systems of differential equations:

794. $\dfrac{dt}{t} = \dfrac{dx}{x} = \dfrac{dy}{ty}$.

795. $\dfrac{dt}{xy} = \dfrac{dx}{yt} = \dfrac{dy}{xt}$.

796. $\dfrac{dx}{y} = -\dfrac{dy}{x} = \dfrac{dp}{q} = -\dfrac{dq}{p}$.

797. $\dfrac{dx}{xt} = \dfrac{dy}{-yt} = \dfrac{dt}{xy}$.

798. $\dfrac{dt}{t^2 - x^2 - y^2} = \dfrac{dx}{2tx} = \dfrac{dy}{2ty}$.

799. $\begin{cases} \dfrac{dx}{dt} = \dfrac{3t - 4y}{2y - 3x}, \\ \dfrac{dy}{dt} = \dfrac{4x - 2t}{2y - 3x}. \end{cases}$

800. $\begin{cases} t\,dx = (t - 2x)\,dt, \\ t\,dy = (tx + ty + 2x - t)\,dt. \end{cases}$

801. $\dfrac{t\,dt}{x^2 - 2xy - x^2} = \dfrac{dx}{x + y} = \dfrac{dy}{x - y}$.

22. Integration of homogeneous linear systems with constant coefficients. Euler's method

A homogeneous linear system with constant coefficients is a system of differential equations of the form

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} a_{ik} x_k(t), \quad i = 1, 2, \ldots, n, \tag{1}$$

where the coefficients a_{ik} are constants and $x_k(t)$ are the desired functions of t.

System (1) may be briefly written in the form of one matrix equation

$$\frac{dX}{dt} = AX, \tag{2}$$

where

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad X=\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \frac{dX}{dt}=\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix},$$

The column matrix

$$Y(t)=\begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}$$

is said to be *a particular solution* of equation (2) in the interval (a, b) if the identity

$$\frac{dY}{dt}=AY(t)$$

holds for $a<t<b$.

The system of particular solutions

$$X_1(t)=\begin{pmatrix} x_1^{(1)}(t) \\ x_1^{(2)}(t) \\ \vdots \\ x_1^{(n)}(t) \end{pmatrix}, \quad X_2(t)=\begin{pmatrix} x_2^{(1)}(t) \\ x_2^{(2)}(t) \\ \vdots \\ x_2^{(n)}(t) \end{pmatrix},$$

$$\dots, \quad X_n(t)=\begin{pmatrix} x_n^{(1)}(t) \\ x_n^{(2)}(t) \\ \vdots \\ x_n^{(n)}(t) \end{pmatrix}$$

(the subscript of the notation x_i^k indicating the number of a solution and the superscript that of a function in the solution) is said to be *fundamental* in the interval (a, b) if its Wronskian

$$W(t) \equiv W(X_1, X_2, \dots, X_n)=\begin{vmatrix} x_1^{(1)}(t) & x_2^{(1)}(t) & \dots & x_n^{(1)}(t) \\ x_1^{(2)}(t) & x_2^{(2)}(t) & \dots & x_n^{(2)}(t) \\ \dots & \dots & \dots & \dots \\ x_1^{(n)}(t) & x_2^{(n)}(t) & \dots & x_n^{(n)}(t) \end{vmatrix} \neq 0$$

for all $t \in (a, b)$.

Theorem. *If the system of particular solutions of the homogeneous equation* (2) *is fundamental, then the general solution of the equation is of the form*

$$X(t) = C_1X_1(t) + C_2X_2(t) + \ldots + C_nX_n(t),$$

$C_1, C_2, \ldots, C_n$ being arbitrary constants.

Linear systems can be integrated by the various methods considered before, for example by the method of elimination, by finding integrable combinations, etc.

Euler's method is also used to integrate homogeneous linear systems with constant coefficients.

We shall consider this method in application to a system of three linear differential equations:

$$\begin{cases} \dfrac{dx}{dt} = ax + by + cz, \\ \dfrac{dy}{dt} = a_1x + b_1y + c_1z, \\ \dfrac{dz}{dt} = a_2x + b_2y + c_2z. \end{cases} \tag{3}$$

We seek the solution of system (3) in the form

$$x = \lambda e^{rt}, \quad y = \mu e^{rt}, \quad z = \nu e^{rt}, \quad \text{with } \lambda, \mu, \nu, \text{ and } r \text{ constant.} \tag{4}$$

Substituting (4) in (3) and cancelling e^{rt} we obtain a system of equations from which we can determine λ, μ, and ν:

$$\begin{cases} (a-r)\lambda + b\mu + c\nu = 0, \\ a_1\lambda + (b_1 - r)\mu + c_1\nu = 0, \\ a_2\lambda + b_2\mu + (c_2 - r)\nu = 0. \end{cases} \tag{5}$$

System (5) has a nonzero solution when its determinant is zero

$$\Delta = \begin{vmatrix} a-r & b & c \\ a_1 & b_1 - r & c_1 \\ a_2 & b_2 & c_2 - r \end{vmatrix} = 0. \tag{6}$$

Equation (6) is called the *characteristic* equation.

A. Let the roots r_1, r_2, and r_3 of the characteristic equation be real and distinct. By substituting r_1 for r in (5) and solving system (5) we obtain the numbers λ_1, μ_1, and ν_1. We

then set $r = r_2$ in (5) and get the numbers λ_2, μ_2, ν_2 and finally we get λ_3, μ_3, ν_3 for $r = r_3$. We obtain three particular solutions corresponding to the three sets of numbers λ, μ, and ν

$$x_1 = \lambda_1 e^{r_1 t}, \quad y_1 = \mu_1 e^{r_1 t}, \quad z_1 = \nu_1 e^{r_1 t},$$
$$x_2 = \lambda_2 e^{r_2 t}, \quad y_2 = \mu_2 e^{r_2 t}, \quad z_2 = \nu_2 e^{r_2 t},$$
$$x_3 = \lambda_3 e^{r_3 t}, \quad y_3 = \mu_3 e^{r_3 t}, \quad z_3 = \nu_3 e^{r_3 t}.$$

The general solution of system (3) is of the form

$$x = C_1\lambda_1 e^{r_1 t} + C_2\lambda_2 e^{r_2 t} + C_3\lambda_3 e^{r_3 t},$$
$$y = C_1\mu_1 e^{r_1 t} + C_2\mu_2 e^{r_2 t} + C_3\mu_3 e^{r_3 t},$$
$$z = C_1\nu_1 e^{r_1 t} + C_2\nu_2 e^{r_2 t} + C_3\nu_3 e^{r_3 t}.$$

Example 1. Solve the system

$$\begin{cases} \dfrac{dx}{dt} = 3x - y + z, \\ \dfrac{dy}{dt} = -x + 5y - z, \\ \dfrac{dz}{dt} = x - y + 3z. \end{cases}$$

Solution. We set up the characteristic equation

$$\begin{vmatrix} 3-r & -1 & 1 \\ -1 & 5-r & -1 \\ 1 & -1 & 3-r \end{vmatrix} = 0,$$

or $r^3 - 11r^2 + 36r - 36 = 0$.

The following numbers correspond to the roots $r_1 = 2$, $r_2 = 3$, $r_3 = 6$

$$\lambda_1 = 1, \quad \mu_1 = 0, \quad \nu_1 = -1;$$
$$\lambda_2 = 1, \quad \mu_2 = 1, \quad \nu_2 = 1;$$
$$\lambda_3 = 1, \quad \mu_3 = -2, \quad \nu_3 = 1.$$

We write out the particular solutions

$$x_1 = e^{2t}, \quad y_1 = 0, \quad z_1 = -e^{2t},$$
$$x_2 = e^{3t}, \quad y_2 = e^{3t}, \quad z_2 = e^{3t},$$
$$x_3 = e^{6t}, \quad y_3 = -2e^{6t} \quad z_3 = e^{6t}.$$

The general solution of the system is

$$x = C_1 e^{2t} + C_2 e^{3t} + C_3 e^{6t},$$
$$y = C_2 e^{3t} - 2C_3 e^{6t},$$
$$z = -C_1 e^{2t} + C_2 e^{3t} + C_3 e^{6t}.$$

B. Now consider the case where the characteristic equation has complex roots.

Example 2. Solve the system

$$\begin{cases} \frac{dx}{dt} = x - 5y, \\ \frac{dy}{dt} = 2x - y. \end{cases} \tag{7}$$

Solution. We write out a system to determine λ and μ

$$\begin{cases} (1-r)\lambda - 5\mu = 0, \\ 2\lambda - (1+r)\mu = 0. \end{cases} \tag{8}$$

The characteristic equation

$$\begin{vmatrix} 1-r & -5 \\ 2 & -1-r \end{vmatrix} = 0$$

has the roots $r_1 = 3i$, $r_2 = -3i$. Substituting $r_1 = 3i$ in (8) we obtain two equations to determine λ_1 and μ_1:

$$(1 - 3i)\lambda_1 - 5\mu_1 = 0, \qquad 2\lambda_1 - (1 + 3i)\mu_1 = 0$$

one of which is a consequence of the other (by virtue of the fact that the determinant of system (8) is equal to zero).

We take $\lambda_1 = 5$, $\mu_1 = 1 - 3i$, then the first particular solution will be written as

$$x_1 = 5e^{3it}, \qquad y_1 = (1 - 3i)e^{3it}. \tag{9}$$

Similarly, substituting the root $r_2 = -3i$ in (8), we find the second particular solution:

$$x_2 = 5e^{-3it}, \qquad y_2 = (1 + 3i)e^{-3it}. \tag{10}$$

We go over to a new fundamental system of solutions:

$$\begin{aligned} \tilde{x}_1 &= \frac{x_1 + x_2}{2}, \quad \tilde{x}_2 = \frac{x_1 - x_2}{2i}, \\ \tilde{y}_1 &= \frac{y_1 + y_2}{2}, \quad \tilde{y}_2 = \frac{y_1 - y_2}{2i}. \end{aligned} \tag{11}$$

From (9), (10), and (11), using the well-known Euler formula $e^{\pm \alpha i t} = \cos \alpha t \pm i \sin \alpha t$, we get

$$\tilde{x}_1 = 5 \cos 3t, \qquad \tilde{x}_2 = 5 \sin 3t,$$

$$\tilde{y}_1 = \cos 3t + 3 \sin 3t, \qquad \tilde{y}_2 = \sin 3t - 3 \cos 3t.$$

The general solution of system (7) is

$$x = C_1\tilde{x}_1 + C_2\tilde{x}_2 = 5C_1 \cos 3t + 5C_2 \sin 3t,$$

$$y = C_1\tilde{y}_1 + C_2\tilde{y}_2 = C_1 (\cos 3t + 3 \sin 3t) + C_2 (\sin 3t - 3 \cos 3t).$$

Remark. Having found the first particular solution (9), we could have at once written the general solution of system (7) using the formulas

$$x = C_1 \operatorname{Re} x_1 + C_2 \operatorname{Im} x_1, \qquad y = C_1 \operatorname{Re} y_1 + C_2 \operatorname{Im} y_1,$$

where $\operatorname{Re} z$ and $\operatorname{Im} z$ denote respectively the real and the imaginary part of a complex number z, i.e. if $z = a + bi$, then $\operatorname{Re} z = a$, $\operatorname{Im} z = b$.

C. The case of multiple roots.

Example 3. Solve the system

$$\begin{cases} \dfrac{dx}{dt} = 2x + y, \\ \dfrac{dy}{dt} = 4y - x. \end{cases} \tag{12}$$

Solution. The characteristic equation

$$\begin{vmatrix} 2-r & 1 \\ -1 & 4-r \end{vmatrix} = 0 \text{ or } r^2 - 6r + 9 = 0$$

has the multiple root $r_1 = r_2 = 3$.

The solution should be sought in the form

$$x = (\lambda_1 + \mu_1 t)\, e^{3t}, \quad y = (\lambda_2 + \mu_2 t)\, e^{3t}. \tag{13}$$

Substituting (13) in the first equation of system (12) we get

$$3 (\lambda_1 + \mu_1 t) + \mu_1 = 2 (\lambda_1 + \mu_1 t) + (\lambda_2 + \mu_2 t). \tag{14}$$

Comparing the coefficients of equal powers of t on the left and right of (14) we get

$$3\lambda_1 + \mu_1 = 2\lambda_1 + \lambda_2,$$
$$3\mu_1 = 2\mu_1 + \mu_2,$$

whence

$$\lambda_2 = \lambda_1 + \mu_1, \quad \mu_2 = \mu_1. \tag{15}$$

The quantities λ_1 and μ_1 remain arbitrary. Denoting them by C_1 and C_2 respectively, we obtain the general solution of system (12):

$$x = (C_1 + C_2 t)\, e^{3t}, \qquad y = (C_1 + C_2 + C_2 t)\, e^{3t}.$$

Remark. It is easily verified that if we substitute (13) in the second equation of system (12), we shall get the same result (15). Indeed, from the equation

$$\mu_2 + 3(\lambda_2 + \mu_2 t) = 4(\lambda_2 + \mu_2 t) - (\lambda_1 + \mu_1 t)$$

we obtain two relations to determine λ_2 and μ_2 in terms of λ_1 and μ_1:

$$\mu_2 + 3\lambda_2 = 4\lambda_2 - \lambda_1,$$
$$3\mu_2 = 4\mu_2 - \mu_1,$$

whence $\lambda_2 = \lambda_1 + \mu_2$, $\mu_2 = \mu_1$.

Example 4. Solve the Cauchy problem for the system

$$\begin{cases} \dfrac{dx}{dt} = 8y, \\ \dfrac{dy}{dt} = -2z, \\ \dfrac{dz}{dt} = 2x + 8y - 2z, \end{cases} \tag{16}$$

the initial conditions being $x(0) = -4$, $y(0) = 0$, $z(0) = 1$.

Solution. The characteristic equation is

$$\begin{vmatrix} -r & 8 & 0 \\ 0 & -r & -2 \\ 2 & 8 & -2-r \end{vmatrix} = 0 \text{ or } (r+2)(r^2+16) = 0. \tag{17}$$

The roots of equation (17) are $r_1 = -2$, $r_2 = 4i$, $r_3 = -4i$. Corresponding to the real root $r_1 = -2$ is the solution

$$x_1 = \lambda_1 e^{-2t}, \quad y_1 = \mu_1 e^{-2t}, \quad z_1 = \nu_1 e^{-2t}. \tag{18}$$

Substituting (18) in system (16) and cancelling e^{-2t} we get

$$-2\lambda_1 = 8\mu_1, \quad -2\mu_1 = -2\nu_1, \quad -2\nu_1 = 2\lambda_1 + 8\mu_1 - 2\nu_1,$$

whence $\lambda_1 = -4\mu_1$, $\nu_1 = \mu_1$. We set $\mu_1 = 1$, for example, then $\lambda_1 = -4$, $\nu_1 = 1$ and the particular solution (18) becomes

$$x_1 = -4e^{-2t}, \quad y_1 = e^{-2t}, \quad z_1 = e^{-2t}. \tag{19}$$

Corresponding to the complex root $r_2 = 4i$ is the solution

$$x_2 = \lambda_2 e^{4it}, \quad y_2 = \mu_2 e^{4it}, \quad z_2 = \nu_2 4^{4it}$$

by substituting which in (16) and cancelling e^{4it} we get

$$4i\lambda_2 = 8\mu_2, \quad 4i\mu_2 = -2\nu_2, \quad 4i\nu_2 = 2\lambda_2 + 8\mu_2 - 2\nu_2,$$

whence $\lambda_2 = -2i\mu_2$, $\nu_2 = -2i\mu_2$, so that for $\mu_2 = i$, for example, we have $\lambda_2 = 2$, $\nu_2 = 2$ and the particular solution is

$$x_2 = 2\, e^{4it}, \quad y_2 = i\, e^{4it}, \quad z_2 = 2e^{4it}. \tag{20}$$

Corresponding to the root $r_3 = -4i$ is a solution complex conjugate to solution (20), i.e.

$$x_3 = 2\, e^{-4it}, \quad y_3 = -ie^{-4it}, \quad z_3 = 2\, e^{-4it}. \tag{21}$$

Taking into account (19), (20), (21) we obtain the general solution

$$\begin{aligned} x &= -4C_1 e^{-2t} + 2C_2\, e^{4it} + 2C_3\, e^{-4it}, \\ y &= C_1\, e^{-2t} + C_2 i\, e^{4it} - C_3 i\, e^{-4it}, \\ z &= C_1\, e^{-2t} + 2C_2\, e^{4it} + 2C_3\, e^{-4it}. \end{aligned} \tag{22}$$

Finally, we separate out the solution with the initial conditions $x(0) = -4$, $y(0) = 0$, $z(0) = 1$.

When $t = 0$ (22) gives

$$\begin{cases} -4 = -4C_1 + 2C_2 + 2C_3, \\ 0 = C_1 + C_2 i - C_3 i, \\ 1 = C_1 + 2C_2 + 2C_3, \end{cases}$$

whence $C_1 = 1$, $C_2 = i/2$, $C_3 = -i/2$:

thus

$$x=-4e^{-2t}+ie^{4it}-ie^{-4it},$$

$$y=e^{-2t}-\frac{1}{2}e^{i4t}-\frac{1}{2}e^{-i4t},$$

$$z=e^{-2t}+ie^{i4t}-ie^{-4it}.$$

By taking advantage of the Euler formulas $e^{\pm\alpha it}=\cos\alpha t \pm i\sin\alpha t$ we ultimately get

$$x=-4e^{-2t}-2\sin 4t, \quad y=e^{-2t}-\cos 4t,$$

$$z=e^{-2t}-2\sin 4t.$$

Find the general solution of the given systems by Euler's method and separate out the solution satisfying the given initial conditions, where indicated.

802. $\begin{cases} \dfrac{dx}{dt}=8y-x, \\ \dfrac{dy}{dt}=x+y. \end{cases}$

803. $\begin{cases} \dfrac{dx}{dt}=x-y, \\ \dfrac{dy}{dt}=y-x. \end{cases}$

804. $\begin{cases} \dfrac{dx}{dt}=2x+y, \\ \dfrac{dy}{dt}=x-3y, \quad x(0)=y(0)=0. \end{cases}$

805. $\begin{cases} \dfrac{dx}{dt}=x+y, \\ \dfrac{dy}{dt}=4y-2x, \quad x(0)=0, \quad y(0)=-1. \end{cases}$

806. $\begin{cases} \dfrac{dx}{dt}=4x-5y, \\ \dfrac{dy}{dt}=x, \quad x(0)=0, \quad y(0)=1. \end{cases}$

807. $\begin{cases} \dfrac{dx}{dt}=-x+y+z, \\ \dfrac{dy}{dt}=x-y+z, \\ \dfrac{dz}{dt}=x+y-z. \end{cases}$

808. $\begin{cases} \dfrac{dx}{dt} = 2x - y + z, \\ \dfrac{dy}{dt} = x + 2y - z, \\ \dfrac{dz}{dt} = x - y + 2z. \end{cases}$

809. $\begin{cases} \dfrac{dx}{dt} = 2x - y + z, \\ \dfrac{dy}{dt} = x + z, \\ \dfrac{dz}{dt} = y - 2z - 3x, \quad x(0) = 0, \quad y(0) = 0, \quad z(0) = 1. \end{cases}$

23. Methods of integrating nonhomogeneous linear systems with constant coefficients

Consider the nonhomogeneous linear system with constant coefficients

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} a_{ik} x_k(t) + f_i(t), \quad i = 1, 2, \ldots, n$$

which can be briefly written in the matrix form

$$\frac{dX}{dt} = AX + F,$$

where F is a column matrix whose entries are functions $f_i(t)$.

Theorem. *The general solution $X(t)$ of a nonhomogeneous linear system is the sum of the general solution $X_{g \cdot h}(t)$ of the corresponding homogeneous system $\frac{dX}{dt} = AX$ and any particular solution $X_{p \cdot n}(t)$ of the given nonhomogeneous system*

$$X(t) = X_{g \cdot h}(t) + X_{p \cdot n}(t) = \sum_{k=1}^{n} C_k X_k(t) + X_{p \cdot n}(t),$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

We shall consider some methods of integrating nonhomogeneous linear systems.

23.1. The method of variation of arbitrary parameters (the Lagrange method). As an illustration of how this

method can be applied we shall consider a system of three nonhomogeneous equations. Let the following system be given

$$\begin{cases} x' + a_1x + b_1y + e_1z = f_1(t), & (1.1) \\ y' + a_2x + b_2y + e_2z = f_2(t), & (1.2) \\ z' + a_3x + b_3y + e_3z = f_3'(t). & (1.3) \end{cases} \tag{1}$$

We shall assume that the general solution of the corresponding homogeneous system has already been found:

$$\begin{aligned} x &= C_1x_1 + C_2x_2 + C_3x_3, \\ y &= C_1y_1 + C_2y_2 + C_3y_3, \\ z &= C_1z_1 + C_2z_2 + C_3z_3. \end{aligned} \tag{2}$$

We seek the solution of the nonhomogeneous system (1) in the form

$$\begin{aligned} x &= C_1(t)\,x_1 + C_2(t)\,x_2 + C_3(t)\,x_3, \\ y &= C_1(t)\,y_1 + C_2(t)\,y_2 + C_3(t)\,y_3, \\ z &= C_1(t)\,z_1 + C_2(t)\,z_2 + C_3(t)\,z_3, \end{aligned} \tag{3}$$

$C_1(t)$, $C_2(t)$, $C_3(t)$ being as yet unknown functions.

We substitute (3) in (1); then equation (1.1) becomes

$$\begin{aligned} C_1'x_1 + C_2'x_2 + C_3'x_3 + C_1(x_1' + a_1x_1 + b_1y_1 + e_1z_1) + \\ + C_2(x_2' + a_1x_2 + b_1y_2 + e_1z_2) + C_3(x_3' + a_1x_3 + \\ + b_1y_3 + e_1z_3) = f_1(t). \end{aligned} \tag{4}$$

All the sums in parentheses will vanish (due to the fact that (2) is the solution of the corresponding homogeneous system), so that we shall have

$$C_1'x_1 + C_2'x_2 + C_3'x_3 = f_1(t). \tag{5}$$

Similarly, on substituting (3) in (1.2) and (1.3) we get

$$\begin{aligned} C_1'y_1 + C_2'y_2 + C_3'y_3 &= f_2(t), \\ C_1'z_1 + C_2'z_2 + C_3'z_3 &= f_3(t). \end{aligned} \tag{6}$$

The system of equations (5), (6), linear in C_1', C_2', C_3', has a solution, for its determinant

$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \neq 0$$

by virtue of the particular solutions of the corresponding homogeneous system being linearly independent. After finding $C_1'(t)$, $C_2'(t)$, $C_3'(t)$ we find, by integrating, $C_1(t)$, $C_2(t)$, $C_3(t)$ and thereby solution (3) of the nonhomogeneous system (1).

Example 1. Using the method of variation of parameters olve the system

$$\begin{cases} \frac{dx}{dt} + 2x + 4y = 1 + 4t, \\ \frac{dy}{dt} + x - y = \frac{3}{2}t^2. \end{cases} \tag{7}$$

Solution. We first solve the corresponding homogeneous system

$$\begin{cases} \frac{dx}{dt} + 2x + 4y = 0, \\ \frac{dy}{dt} + x - y = 0. \end{cases} \tag{8}$$

From the second equation of system (8) we have

$$x = y - \frac{dy}{dt}, \text{ so that } \frac{dx}{dt} = \frac{dy}{dt} - \frac{d^2y}{dt^2}.$$

Substituting these expressions for x and $\frac{dx}{dt}$ in the first equation of system (8) we get

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 0;$$

the general solution of this equation is

$$y = C_1e^{2t} + C_2e^{-3t}.$$

Since $x = y - \frac{dy}{dt}$, we have

$$x = -C_1e^{2t} + 4C_2e^{-3t}.$$

The general solution of the homogeneous system (8) is

$$x = -C_1\,e^{2t} + 4C_2\,e^{-3t}, \quad y = C_1\,e^{2t} + C_2\,e^{-3t}.$$

We seek the solution of the nonhomogeneous system (7) in the form

$$x = -C_1(t)\,e^{2t} + 4C_2(t)\,e^{-3t}, \quad y = C_1(t)\,e^{2t} + C_2(t)\,e^{-3t}. \tag{9}$$

Substituting (9) in (7) and gathering together the similar terms we get

$$\begin{cases} -C_1'(t)\,e^{2t}+4C_2'(t)\,e^{-3t}=1+4t, \\ C_1'(t)\,e^{2t}+C_2'(t)\,e^{-3t}=\dfrac{3}{2}t^2, \end{cases}$$

whence

$$C_1'(t)=\frac{(6t^2-4t-1)\,e^{-2t}}{5}, \quad C_2'(t)=\frac{(3t^2+8t+2)\,e^{3t}}{10}.$$

Integrating we get

$$C_1(t)=-\frac{1}{5}(t+3t^2)\,e^{-2t}+C_1,$$

$$C_2(t)=\frac{1}{10}(2t+t^2)\,e^{3t}+C_2, \tag{10}$$

where C_1 and C_2 are arbitrary constants. Substituting (10) in (9) we obtain the general solution of system (7)

$$x=-C_1e^{2t}+4C_2e^{-3t}+t+t^2, \quad y=C_1e^{2t}+C_2e^{-3t}-\frac{1}{2}t^2.$$

Find the general solution of the nonhomogeneous linear equations below by the method of variation of arbitrary parameters:

810. $\begin{cases} \dfrac{dx}{dt}+2x-y=-e^{2t}, \\ \dfrac{dy}{dt}+3x-2y=6e^{2t}. \end{cases}$

811. $\begin{cases} \dfrac{dx}{dt}=x+y-\cos t, \\ \dfrac{dy}{dt}=-y-2x+\cos t+\sin t, \quad x(0)=1, \quad y(0)=-2. \end{cases}$

812. $\begin{cases} \dfrac{dx}{dt}=y+\tan^2 t-1, \\ \dfrac{dy}{dt}=\tan t-x. \end{cases}$

813. $\begin{cases} \dfrac{dx}{dt}=-4x-2y+\dfrac{2}{e^t-1}, \\ \dfrac{dy}{dt}=6x+3y-\dfrac{3}{e^t-1}. \end{cases}$

814. $\begin{cases} \dfrac{dx}{dt}=y, \\ \dfrac{dy}{dt}=-x+\dfrac{1}{\cos t}. \end{cases}$

23.2. The method of undetermined coefficients (trial and error method). This method is applied to the solution of a nonhomogeneous system of linear equations when functions $f_i(t)$ on the right of the system are of a special form: polynomials $P_k(t)$, exponential functions $e^{\alpha t}$, sines and cosines $\sin\beta t$, $\cos\beta t$ and the products of these functions. A particular solution of the nonhomogeneous system, x_{kr}, is found proceeding from the form of the right-hand side of the system (see Table 1 of Sec. 16.3).

Example 2. Find the general solution of the nonhomogeneous system

$$\begin{cases} \dfrac{dx}{dt} = x - 2y + e^t, \\ \dfrac{dy}{dt} = x + 4y + e^{2t}. \end{cases} \tag{11}$$

Solution. We first find the general solution of the corresponding homogeneous system

$$\begin{cases} \dfrac{dx}{dt} = x - 2y, \\ \dfrac{dy}{dt} = x + 4y. \end{cases} \tag{12}$$

The characteristic equation is of the form

$$\begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - 5\lambda + 6 = 0.$$

The roots of the equation are $\lambda_1 = 2$, $\lambda_2 = 3$. Corresponding to the root $\lambda_1 = 2$ is the particular solution of the system

$$x_1 = \mu_1 e^{2t}, \quad y_1 = \nu_1 e^{2t}.$$

Substituting x_1 and y_1 in (12) we obtain a system of equations to find μ_1 and ν_1:

$$-\mu_1 - 2\nu_1 = 0, \quad \mu_1 + 2\nu_1 = 0.$$

Hence we have, for example, $\mu_1 = 2$, $\nu_1 = -1$, so that the first particular solution of the homogeneous system (11) is

$$x_1 = 2e^{2t}, \quad y_1 = -e^{2t}.$$

Corresponding to the root $\lambda_2 = 3$ is the particular solution

$$x_2 = \mu_2 e^{3t}, \quad y_2 = \nu_2 e^{3t}.$$

We find the numbers μ_2 and ν_2 from the system

$$\begin{cases} -2\mu_2-2\nu_2=0, \\ \mu_2+\nu_2=0, \end{cases}$$

which is satisfied, for example, by the numbers $\mu_2 = 1$, $\nu_2 = -1$. Then the second particular solution of system (12) is

$$x_2 = e^{3t}, \quad y_2 = -e^{3t}.$$

The general solution of the homogeneous system (12) is

$$\tilde{x} = 2C_1 e^{2t} + C_2 e^{3t}, \quad \tilde{y} = -C_1 e^{2t} - C_2 e^{3t}.$$

Using the method of undetermined coefficients we find the particular solution of system (11). Taking into account the form of the right-hand sides $f_1(t) = e^t$ and $f_2(t) = e^{2t}$ we write the form of the particular solution (see Table 1)

$$x_{\text{p.n}} = K e^t + (Lt + M) e^{2t}, \quad y_{\text{p. n}} = N e^t + (Pt + Q) e^{2t}. \tag{13}$$

Substituting (13) in (11) we have

$$K e^t + 2(Lt + M) e^{2t} + L e^{2t} = K e^t + (Lt + M) e^{2t} - 2N e^t - 2(Pt + Q) e^{2t} + e^t,$$

$$N e^t + 2(Pt + Q) e^{2t} + P e^{2t} = K e^t + (Lt + M) e^{2t} + 4N e^t + 4(Pt + Q) e^{2t} + e^{2t}.$$

Equating the coefficients of e^t, e^{2t}, and te^{2t} on both sides of these identities we get

$$\begin{array}{l|l} e^t & K=K-2N+1, \\ e^{2t} & 2M+L=M-2Q, \\ te^{2t} & 2L=L-2P, \end{array}$$

from the first identity and

$$\begin{array}{l|l} e^t & N=K+4N, \\ e^{2t} & 2Q+P=M+4Q+1, \\ te^{2t} & 2P=L+4P \end{array}$$

from the second. Solving this system of equations we obtain $K = -3/2$, $L = 2$, $M = 0$, $N = 1/2$, $P = -1$, $Q = -1$.

So the particular solution (13) is of the form

$$x_{\text{p}\cdot\text{n}} = -\frac{3}{2}\,e^{t} + 2t\,e^{2t}, \quad y_{\text{p}\cdot\text{n}} = \frac{1}{2}\,e^{t} - (t+1)\,e^{2t}.$$

The general solution of the nonhomogeneous system is

$$x = 2C_1\,e^{2t} + C_2\,e^{3t} - \frac{3}{2}\,e^{t} + 2t\,e^{2t},$$

$$y = -C_1\,e^{2t} - C_2\,e^{3t} + \frac{1}{2}\,e^{t} - (t+1)\,e^{2t}.$$

Example 3. Solve the system

$$\begin{cases} \dfrac{dx}{dt} = x + 2y, \\ \dfrac{dy}{dt} = x - 5\sin t. \end{cases} \tag{14}$$

Solution. The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - \lambda - 2 = 0.$$

The roots of the characteristic equation are $\lambda_1 = -1$, $\lambda_2 = 2$. The general solution of the corresponding homogeneous system is

$$\tilde{x} = C_1\,e^{-t} + 2C_2\,e^{2t}, \quad \tilde{y} = -C_1\,e^{-t} + C_2\,e^{2t}.$$

We find the particular solution of the nonhomogeneous system (14) keeping in mind that $f_1(t) = 0$, $f_2(t) = -5\sin t$. We write $x_{\text{p}\cdot\text{n}}$ and $y_{\text{p}\cdot\text{n}}$ in the form

$$x_{\text{p}\cdot\text{n}} = A\cos t + B\sin t, \quad y_{\text{p}\cdot\text{n}} = M\cos t + N\sin t$$

and substitute into system (14):

$$-A\sin t + B\cos t = A\cos t + B\sin t + 2M\cos t + 2N\sin t,$$

$$-M\sin t + N\cos t = A\cos t + B\sin t - 5\sin t.$$

We equate the coefficients of $\sin t$ and $\cos t$ on both sides of the equations:

$$\begin{cases} -A = B + 2N, \\ B = A + 2M, \\ -M = B - 5, \\ N = A, \end{cases}$$

hence $A = -1$, $B = 3$, $M = 2$, $N = -1$, so that

$$x_{\text{p}\cdot\text{n}} = -\cos t + 3\sin t, \quad y_{\text{p}\cdot\text{n}} = 2\cos t - \sin t.$$

The general solution of the original system is

$$x = \tilde{x} + x_{\text{p}\cdot\text{n}} = C_1 e^{-t} + 2C_2 e^{2t} - \cos t + 3\sin t,$$

$$y = \tilde{y} + y_{\text{p}\cdot\text{n}} = -C_1 e^{-t} + C_2 e^{2t} + 2\cos t - \sin t.$$

Example 4. Solve the system

$$\begin{cases} \dfrac{dx}{dt} = x + 2y + 16t\,e^t, \\ \dfrac{dy}{dt} = 2x - 2y. \end{cases} \tag{15}$$

Solution. The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 + \lambda - 6 = 0.$$

The roots of the characteristic equation are $\lambda_1 = 2$, $\lambda_2 = -3$. The general solution of the homogeneous system corresponding to system (15) is

$$\tilde{x} = 2C_1 e^{2t} + C_2 e^{-3t}, \quad \tilde{y} = C_1 e^{2t} - 2C_2 e^{-3t}.$$

The particular solution of the nonhomogeneous equation (15) is sought in the form

$$x_{\text{p}\cdot\text{n}} = (At + B)\,e^t, \quad y_{\text{p}\cdot\text{n}} = (Mt + N)\,e^t. \tag{16}$$

We substitute (16) in (15) and cancel e^t:

$$At + B + A = At + B + 2Mt + 2N + 16t,$$

$$Mt + N + M = 2At + 2B - 2Mt - 2N,$$

hence $A = -12$, $B = -13$, $M = -8$, $N = -6$; so

$$x_{\text{p}\cdot\text{n}} = -(12t + 13)\,e^t, \quad y_{\text{p}\cdot\text{n}} = -(8t + 6)\,e^t.$$

The general solution of the original system is

$$x = \tilde{x} + x_{\text{p}\cdot\text{n}} = 2C_1 e^{2t} + C_2 e^{-3t} - (12t + 13)\,e^t,$$

$$y = \tilde{y} + y_{\text{p}\cdot\text{n}} = C_1 e^{2t} - 2C_2 e^{-3t} - (8t + 6)\,e^t. \ \blacklozenge$$

Integrate the nonhomogeneous linear systems with constant coefficients:

815. $\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = 1 - x. \end{cases}$

816. $\begin{cases} \dfrac{dx}{dt} = 3 - 2y, \\ \dfrac{dy}{dt} = 2x - 2t. \end{cases}$

817. $\begin{cases} \dfrac{dx}{dt} = -y + \sin t, \\ \dfrac{dy}{dt} = x + \cos t. \end{cases}$

818. $\begin{cases} \dfrac{dx}{dt} = x + y + e^t, \\ \dfrac{dy}{dt} = x + y - e^t. \end{cases}$

819. $\begin{cases} \dfrac{dx}{dt} = 4x - 5y + 4t - 1, \\ \dfrac{dy}{dt} = x - 2y + t, \quad x(0) = y(0) = 0. \end{cases}$

820. $\begin{cases} \dfrac{dx}{dt} = y - x + e^t, \\ \dfrac{dy}{dt} = x - y + e^t, \quad x(0) = y(0) = 1. \end{cases}$

821. $\begin{cases} \dfrac{dx}{dt} + y = t^2, \\ \dfrac{dy}{dt} - x = t. \end{cases}$

822. $\begin{cases} \dfrac{dx}{dt} + \dfrac{dy}{dt} + y = e^{-t}, \\ 2\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2y = \sin t. \end{cases}$

823. $\begin{cases} \dfrac{dx}{dt} = 2x + y - 2z + 2 - t, \\ \dfrac{dy}{dt} = 1 - x, \\ \dfrac{dz}{dt} = x + y - z + 1 - t. \end{cases}$

824. $\begin{cases} \dfrac{dx}{dt}+x+2y=2e^{-t}, \\ \dfrac{dy}{dt}+y+z=1, \\ \dfrac{dz}{dt}+z=1, \quad x(0)=y(0)=z(0)=1. \end{cases}$

23.3. Constructing integrable combinations (d'Alembert's method). This method is used for constructing integrable combinations in solving systems of linear equations with constant coefficients. We shall show how it can be applied to the solution of systems of two equations:

$$\begin{cases} \dfrac{dx}{dt}=a_1x+b_1y+f_1(t), \\ \dfrac{dy}{dt}=a_2x+b_2y+f_2(t). \end{cases} \tag{17}$$

Multiply the second equation by some number λ and add termwise to the first equation:

$$\frac{d(x+\lambda y)}{dt}=(a_1+\lambda a_2)x+(b_1+\lambda b_2)y+f_1(t)+\lambda f_2(t).$$

Rewrite this last equation in the form

$$\frac{d(x+\lambda y)}{dt}=(a_1+\lambda a_2)\left(x+\frac{b_1+\lambda b_2}{a_1+\lambda a_2}y\right)+f_1(t)+\lambda f_2(t). \tag{18}$$

Choose the number λ so that

$$\frac{b_1+\lambda b_2}{a_1+\lambda a_2}=\lambda. \tag{19}$$

Then (18) is reduced to an equation linear in $x+\lambda y$

$$\frac{d(x+\lambda y)}{dt}=(a_1+\lambda a_2)(x+\lambda y)+f_1(t)+\lambda f_2(t)$$

which, on integrating, gives

$$x+\lambda y=e^{(a_1+\lambda a_2)t}\left\{C+\int [f_1(t)+\lambda f_2(t)]\,e^{-(a_1+\lambda a_2)t}\,dt\right\}. \tag{20}$$

If equation (19) has distinct real roots λ_1 and λ_2, then we obtain two first integrals of system (17) from (20) and so the integration of the system will be completed.

Example 5. Solve the following system by d'Alembert's method

$$\begin{cases} \dfrac{dx}{dt} = 5x + 4y + e^t, \\ \dfrac{dy}{dt} = 4x + 5y + 1. \end{cases} \tag{21}$$

Solution. We choose λ by formula (19): $4 + 5\lambda = \lambda(5 + 4\lambda)$, whence $\lambda_{1,2} = \pm 1$.
Then for the case $\lambda = 1$ formula (20) gives

$$x + y = e^{(5+4\times 1)t}\left\{C_1 + \int (e^t + 1)\, e^{-(5+4\times 1)t}\, dt\right\} =$$

$$= e^{9t}\left\{C_1 + \int (e^{-8t} + e^{-9t})\, dt\right\} = C_1 e^{9t} - \frac{1}{8} e^t - \frac{1}{9}.$$

Similarly for $\lambda = -1$ we get

$$x - y = e^{(5-4)t}\left\{C_2 + \int (e^t - 1)\, e^{-(5-4)t}\, dt\right\} = C_2 e^t + t\, e^t + 1.$$

Thus we have two independent first integrals of system (21):

$$\left(x + y + \frac{1}{8} e^t + \frac{1}{9}\right) e^{-9t} = C_1, \quad (x - y - te^t - 1)\, e^{-t} = C_2.$$

The integration of the system is completed.

Remark. If the right-hand sides of a normal system of equations are of the form $\dfrac{ax + by + cz + P(t)}{t}$, where a, b, c are constants and $P(t)$ is a polynomial of t, then the substitution $t = e^\tau$ leads to a system with constant coefficients.

Example 6. Solve the system of equations

$$\begin{cases} t\dfrac{dx}{dt} = -2x + 2y + t, \\ t\dfrac{dy}{dt} = -x - 5y + t^2. \end{cases}$$

Solution. We first substitute $t = e^\tau$. Then

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{1}{t}\frac{dx}{d\tau}; \quad \frac{dy}{dt} = \frac{1}{t}\frac{dy}{d\tau}$$

and the system takes the form

$$\begin{cases} \dfrac{dx}{d\tau} = -2x + 2y + e^{\tau}, \\ \dfrac{dy}{d\tau} = -x - 5y + e^{2\tau}. \end{cases} \tag{22}$$

In order to solve system (22) we employ d'Alembert's method. We multiply the second equation of the system by λ and add termwise to the first:

$$\frac{d}{d\tau}(x+\lambda y) = (-2-\lambda)\,x + (2-5\lambda)\,y + e^{\tau} + \lambda e^{2\tau}$$

or

$$\frac{d}{d\tau}(x+\lambda y) = (-2-\lambda)\left[x + \frac{2-5\lambda}{-2-\lambda}\,y\right] + e^{\tau} + \lambda e^{2\tau}. \tag{23}$$

We choose λ so that the coefficient of y in square brackets should be equal to λ, i.e. $\frac{2-5\lambda}{-2-\lambda} = \lambda$ or $\lambda^2 - 3\lambda + 2 = 0$, whence $\lambda_1 = 1$, $\lambda_2 = 2$. For $\lambda_1 = 1$ (23) gives

$$\frac{d(x+y)}{d\tau} = -3(x+y) + e^{\tau} + e^{2\tau},$$

whence by formula (20) we have

$$x + y = e^{-3\tau}\left[C_1 + \int (e^{\tau} + e^{2\tau})\, e^{3\tau}\, d\tau\right].$$

After integrating we get

$$x + y = C_1 e^{-3\tau} + \frac{1}{4} e^{\tau} + \frac{1}{5} e^{2\tau}. \tag{24}$$

For $\lambda_2 = 2$ we similarly find from (23) that

$$x + 2y = C_2 e^{-4\tau} + \frac{1}{5} e^{\tau} + \frac{1}{3} e^{2\tau}. \tag{25}$$

Solving system (24), (25) for x and y we obtain the general solution of system (22):

$$x = 2C_1 e^{-3\tau} - C_2 e^{-4\tau} + 0.3\, e^{\tau} + \frac{1}{15} e^{2\tau},$$

$$y = -C_1 e^{-3\tau} + C_2 e^{-4\tau} - 0.05\, e^{\tau} + \frac{2}{15} e^{2\tau}.$$

Returning to the variable t ($e^t = t$) we obtain the general solution of the given system

$$x = \frac{2C_1}{t^3} - \frac{C_2}{t^4} + \frac{3t}{10} + \frac{t^2}{15},$$

$$y = -\frac{C_1}{t^3} + \frac{C_2}{t^4} - \frac{t}{20} + \frac{2t^2}{15}.$$

Solve the following systems of equations by d'Alembert's method:

825. $\begin{cases} \dfrac{dx}{dt} = 5x + 4y, \\ \dfrac{dy}{dt} = x + 2y. \end{cases}$

826. $\begin{cases} \dfrac{dx}{dt} = 6x + y, \\ \dfrac{dy}{dt} = 4x + 3y. \end{cases}$

827. $\begin{cases} \dfrac{dx}{dt} = 2x - 4y + 1, \\ \dfrac{dy}{dt} = -x + 5y. \end{cases}$

828. $\begin{cases} \dfrac{dx}{dt} = 3x + y + e^t, \\ \dfrac{dy}{dt} = x + 3y - e^t. \end{cases}$

829. $\begin{cases} \dfrac{dx}{dt} = 2x + 4y + \cos t, \\ \dfrac{dy}{dt} = -x - 2y + \sin t. \end{cases}$

24. Application of the Laplace transformation to the solution of linear differential equations and systems

24.1. General information about the Laplace transformation.

Original and transform. *An original-function is a complex-valued function $f(t)$ of a real variable t satisfying the following conditions*: (1) $f(t) = 0$ if $t < 0$;

(2) $f(t)$ is integrable in any finite interval of the axis of t;

(3) as t increases the absolute value of the function $f(t)$ increases not faster than some exponential function, i. e. there are numbers $M > 0$ and $s_0 \geqslant 0$ such that for all t

we have

$$|f(t)| \leqslant M e^{s_0 t}. \tag{1}$$

The Laplace transform of an original-function is a function $F(p)$ of a complex variable $p = s + i\sigma$, defined by the equation

$$F(p) = \int_0^{+\infty} f(t) e^{-pt}\,dt \tag{2}$$

for Re $p > s_0$. Condition (3) ensures the existence of integral (2).

Transformation (2) relating to the original $f(t)$ its transform $F(p)$ is called *Laplace transformation*. Here we write $f(t) \doteqdot F(p)$.

The properties of the Laplace transformation. Everywhere in what follows we assume that

$$f(t) \doteqdot F(p), \quad g(t) \doteqdot G(p). \tag{3}$$

I. *The property of linearity. For any complex constants α and β*

$$\alpha f(t) + \beta g(t) \doteqdot \alpha F(p) + \beta G(p). \tag{4}$$

II. *Similarity theorem. For any constant $\alpha > 0$*

$$f(\alpha t) \doteqdot \frac{1}{\alpha} F\left(\frac{p}{\alpha}\right). \tag{5}$$

III. *Differentiation of an original. If $f'(t)$ is an original,* then

$$f'(t) \doteqdot pF(p) - f(0). \tag{6}$$

Generalization. *If $f(t)$ is n times continuously differentiable in* $(0, +\infty)$ *and if $f^{(n)}(t)$ is an original,* then

$$f^{(n)}(t) \doteqdot p^n F(p) - p^{n-1} f(0) - p^{n-2} f'(0) - \ldots - f^{(n-1)}(0). \tag{7}$$

IV. *Differentiating a transform is equivalent to multiplying the original by "minus the independent variable"*, i.e.

$$F'(p) \doteqdot -t(f(t). \tag{8}$$

Generalization:

$$F^{(n)}(p) \doteqdot (-1)^n t^n f(t). \tag{9}$$

V. *Integrating an original reduces to dividing the transform by p*:

$$\int_0^t f(t)\,dt \risingdotseq \frac{F(p)}{p}. \tag{10}$$

VI. *Integrating a transform is equivalent to dividing the original by t*:

$$\int_p^\infty F(p)\,dp \risingdotseq \frac{f(t)}{t} \tag{11}$$

$\left(\text{we assume that the integral } \int_p^\infty F(p)\,dp \text{ converges}\right)$.

VII. *Delay theorem. For any positive number* τ

$$f(t-\tau) \risingdotseq e^{-p\tau}F(p). \tag{12}$$

VIII. *Displacement theorem* (multiplication of an original by an exponential function). *For any complex number* λ

$$e^{\lambda t}f(t) \risingdotseq F(p-\lambda). \tag{13}$$

IX. *Multiplication theorem* (A. Borel). *The product of two transforms* $F(p)$ *and* $G(p)$ *is also a transform, with*

$$F(p)\,G(p) \risingdotseq \int_0^t f(\tau)\,g(t-\tau)\,d\tau. \tag{14}$$

The integral on the right of (14) is called *the convolution of the functions* $f(t)$ *and* $g(t)$ and designated by the symbol

$$(f^*g) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau.$$

Theorem IX states that *multiplication of transforms is equivalent to convolution of the originals*, i.e.

$$F(p)\,G(p) \risingdotseq (f^*g). \tag{15}$$

Determining the originals of rational transforms. To find the original $f(t)$ from a known transform $F(p)$, where $F(p) = \frac{A(p)}{B(p)}$ is a proper rational fraction, the following methods are applied.

(i) The fraction is expanded into the sum of partial fractions and the original of each is found using properties I-IX of the Laplace transformation.

(ii) Poles p_k, $k = 1, 2, \ldots, m$, (see [1]) of the fraction and their multiplicities n_k are found. Then the original for $F(p)$ is the function

$$f(t)=\sum_{k=1}^{m}\frac{1}{(n_k-1)!}\lim_{p\to p_k}\frac{\mathrm{d}^{n_k-1}}{\mathrm{d}p^{n_k-1}}\{F(p)(p-p_k)^{n_k}\mathrm{e}^{pt}\}, \quad (16)$$

the sum being taken over all poles of the function $F(t)$.

In case all the poles p_k of the function $F(p)$ are simple, i.e. $n_k = 1$, $k = 1, 2, \ldots, m$, the last formula is simplified and takes the form

$$f(t)=\sum_{k=1}^{m}\frac{A(p_k)}{B'(p_k)}\mathrm{e}^{p_k t}. \quad (17)$$

Example 1. Find the original $f(t)$ if

$$F(p)=\frac{p+2}{(p+1)(p-2)(p^2+4)}.$$

The first way. We represent $F(p)$ as the sum of partial fractions

$$\frac{p+2}{(p+1)(p-2)(p^2+4)}=\frac{A}{p+1}+\frac{B}{p-2}+\frac{Cp+D}{p^2+4}$$

and find the undetermined coefficients A, B, C, D. We have

$$p+2 = A(p-2)(p^2+4)+B(p+1)(p^2+4)$$
$$+(Cp+D)(p+1)(p-2).$$

Setting $p = -1$, $p = 2$, $p = 2i$ in succession in the last equation we get $-15A = 1$, $24B = 4$,

$$(2Ci + D)(2i + 1)(2i - 1) = 2 + 2i,$$

whence $A = -1/15$, $B = 1/6$, $C = -1/10$, $D = -2/5$; so

$$F(p)=-\frac{1}{15}\frac{1}{p+1}+\frac{1}{6}\frac{1}{p-2}-\frac{1}{10}\frac{p+4}{p^2+4}.$$

Finding the original of each of the partial fractions and making use of the property of linearity we get

$$f(t)=-\frac{1}{15}\mathrm{e}^{-t}+\frac{1}{6}\mathrm{e}^{2t}-\frac{1}{10}\cos 2t-\frac{1}{5}\sin 2t.$$

The second way. We find the poles p_k of the function $F(p)$. They coincide with the zeros of the denominator $B(p) = (p+1)(p-2)(p^2+4)$. Thus the transform $F(p)$ has four simple poles $p_1 = -1$, $p_2 = 2$, $p_3 = 2i$, $p_4 = -2i$. Making use of formula (17) we obtain the original

$$f(t) = \sum_{k=1}^{4} \frac{A(p_k)}{B'(p_k)} e^{p_k t} = \sum_{k=1}^{4} \frac{p_k+2}{4p_k^3-3p_k^2+4p-4} e^{p_k t} =$$

$$= -\frac{1}{15} e^{-t} + \frac{1}{6} e^{2t} + \frac{-1+2i}{20} e^{2it} + \frac{-1-2i}{20} e^{-2it} =$$

$$= \frac{1}{6} e^{2t} - \frac{1}{15} e^{-t} - \frac{1}{10} \cos 2t - \frac{1}{5} \sin 2t.$$

Example 2. Find the original $f(t)$ if $F(p) = \dfrac{p+2}{p^3(p-1)^2}$.

Solution. The given fraction $F(p)$ has a pole $p_1 = 0$ of multiplicity $n_1 = 3$ and a pole $p_2 = 1$ of multiplicity $n_2 = 2$. Making use of formula (16) we obtain the original

$$f(t) = \frac{1}{2} \lim_{p\to 0} \frac{d^2}{dp^2} \left[\frac{p+2}{p^3(p-1)^2} p^3 e^{pt} \right] +$$

$$+ \lim_{p\to 1} \frac{d}{dp} \left[\frac{p+2}{p^3(p-1)^2} (p-1)^2 e^{pt} \right] =$$

$$= \frac{1}{2} \lim_{p\to 0} \frac{d^2}{dp^2} \left[\frac{p+2}{(p-1)^2} e^{pt} \right] + \lim_{p\to 1} \frac{d}{dp} \left(\frac{p+2}{p^3} e^{pt} \right) =$$

$$= \frac{1}{2} \lim_{p\to 0} \left\{ \left[\frac{2p+16}{(p-1)^4} - \frac{2t(p+5)}{(p-1)^3} + \frac{t^2(p+2)}{(p-1)^2} \right] e^{pt} \right\} +$$

$$+ \lim_{p\to 1} \left\{ \left[\frac{t(p+2)}{p^3} - \frac{3p^2+5p}{p^4} \right] e^{pt} \right\} = 8 + 5t + t^2 +$$

$$+ (3t-8) e^t.$$

24.2. Solving the Cauchy problem for linear differential equations with constant coefficients. Suppose it is required to find the solution of the second order differential equation with constant coefficients

$$x''(t) + a_1 x'(t) + a_2 x(t) = f(t) \tag{18}$$

satisfying the initial conditions

$$x(0) = x_0, \quad x'(0) = x_1. \tag{19}$$

We shall assume that the function $f(t)$ and the solution $x(t)$ together with its derivatives to the second order inclu-

sively are original-functions. Let $x(t) \risingdotseq X(p), f(t) \risingdotseq F(p)$. By the rule of differentiating originals and taking into account (2) we have

$$x'(t) = pX(p) - x_0, \quad x''(t) = p^2X(p) - px_0 - x_1.$$

Applying the Laplace transformation to both sides of (1) and making use of the property of linearity we obtain the operator equation

$$(p^2 + a_1p + a_2) X(p) = F(p) + x_0(p + a_1) + x_1. \quad (20)$$

Solving equation (20) we find the operator equation

$$X(p) = \frac{F(p) + x_0(p + a_1) + x_1}{p^2 + a_1p + a_2}.$$

Finding the original for $X(p)$ we obtain the solution of equation (18) satisfying the initial conditions (19).

Similarly one can solve any nth order equation with constant coefficients and initial conditions for $t = 0$.

Example 3. Solve the equation

$$x' + x = 1, \quad (21)$$

$$x(0) = 1. \quad (22)$$

Solution. Let $x(t) \risingdotseq X(p)$, then by the rule of differentiating originals we have

$$x'(t) \risingdotseq pX(p) - x(0) = pX(p) - 1.$$

It is known that $1 \risingdotseq 1/p$, therefore going over from the given problem (21), (22) to an operator equation we have

$$pX(p) - 1 + X(p) = 1/p,$$

whence

$$(p + 1) X(p) = 1 + \frac{1}{p} \quad \text{or} \quad X(p) = \frac{1}{p},$$

therefore $x(t) \equiv 1$.

It can easily be seen that the function $x(t) \equiv 1$ satisfies the given equation and the initial condition of the problem.

Example 4. Solve the equation $x'' - 5x' + 4x = 4$, $x(0) = 0$, $x'(0) = 2$.

Solution. Since $4 \risingdotseq 4/p$ and by the condition $x_0 = x(0) = 0$, $x_1 = x'(0) = 2$, the operator equation is of the form $(p^2 - 5p + 4) X(p) = \frac{4}{p} + 2$. Hence we find the operator

solution

$$X(p)=\frac{2p+4}{p(p^2-5p+4)}.$$

We expand the right-hand side into partial fractions:

$$X(p)=\frac{1}{p}-\frac{2}{p-1}+\frac{1}{p-4}.$$

Going over to originals we obtain the desired solution

$$x(t)=1-2e^t+e^{4t}.$$

Example 5. Solve the equation

$$x''+4x'+4x=8e^{-2t},\quad x(0)=1,\quad x'(0)=1.$$

Solution. Since $8e^{-2t} \doteqdot \frac{8}{p+2}$ and by the condition $x_0=x_1=1$, the operator equation is of the form

$$(p^2+4p+4)\,X(p)=\frac{8}{p+2}+p+4+1,$$

and so the operator solution is

$$X(p)=\frac{p^2+7p+18}{(p+2)^3}.$$

We expand the right-hand side into partial fractions:

$$X(p)=\frac{8}{(p+2)^3}+\frac{3}{(p+2)^2}+\frac{1}{p+2}.$$

Going over to the originals we obtain the solution of the problem set

$$x(t)=4t^2e^{-2t}+3te^{-2t}+e^{-2t}.$$

Solve the following equations:

830. $x'+3x=e^{-2t},\quad x(0)=0.$

831. $x'-3x=3t^3+3t^2+2t+1,\quad x(0)=-1.$

832. $x'-x=\cos t-\sin t,\quad x(0)=0.$

833. $2x'+6x=te^{-3t},\quad x(0)=-1/2.$

834. $x'+x=2\sin t,\quad x(0)=0.$

835. $x''=0,\quad x(0)=0,\quad x'(0)=0.$

836. $x''=1,\quad x(0)=0,\quad x'(0)=0.$

837. $x''=\cos t,\quad x(0)=0,\quad x'(0)=0.$

838. $x'' + x' = 0, \quad x(0) = 0, \quad x'(0) = 0.$

839. $x'' + x' = 0, \quad x(0) = 1, \quad x'(0) = -1.$

840. $x'' - x' = 1, \quad x(0) = -1, \quad x'(0) = -1$

841. $x'' + x = t, \quad x(0) = 0, \quad x'(0) = 1.$

842. $x'' + 6x' = 12t + 2, \quad x(0) = 0, \quad x'(0) = 0.$

843. $x'' - 2x' + 2x = 2, \quad x(0) = 1, \quad x'(0) = 0.$

844. $x'' + 4x' + 4x = 4, \quad x(0) = 1, \quad x'(0) = -4.$

845. $2x'' - 2x' = (t + 1)\,e^t, \quad x(0) = 1/2, \quad x'(0) = 1/2.$

846. $x'' + x = 2\cos t, \quad x(0) = -1, \quad x'(0) = 1.$

847. $x'' + 3x' + 2x = 2t^2 + 1, \quad x(0) = 4, \quad x'(0) = -3.$

848. $x'' + x = 2e^t, \quad x(0) = 1, \quad x'(0) = 2.$

849. $x'' - 4x' + 4x = (t - 1)\,e^{2t}, \quad x(0) = 0, \quad x'(0) = 1.$

850. $4x'' - 4x' + x = e^{t/2}, \quad x(0) = -2, \quad x'(0) = 0.$

851. $x'' + 3x' + 2x = e^{-t} + e^{-2t}, \quad x(0) = 2, \quad x'(0) = -3.$

852. $x'' - x' - 6x = 6e^{3t} + 2e^{-2t}, \quad x(0) = 0, \quad x'(0) = 4/5.$

853. $x'' + 4x' + 4x = t^2e^{-2t}, \quad x(0) = x'(0) = 0.$

854. $x'' - x' = 2\sin t, \quad x(0) = 2, \quad x'(0) = 0.$

855. $x'' + 9x = 18\cos 3t, \quad x(0) = 0, \quad x'(0) = 9.$

856. $x'' + 4x = 4\cos 2t - \frac{1}{2}\sin 2t, \quad x(0) = 0, \quad x'(0) = 1/8.$

857. $x'' + 2x' + 3x = t\cos t, \quad x(0) = -1/4, \quad x'(0) = 0.$

858. $x'' - 4x' + 5x = 2e^{2t}(\sin t + \cos t), \quad x(0) = 1, \quad x'(0) = 2.$

859. $x''' - x'' = 0, \quad x(0) = 1, \quad x'(0) = 3, \quad x''(0) = 2.$

860. $x''' - 4x' = 1, \quad x(0) = 0, \quad x'(0) = -1/4, \quad x''(0) = 0.$

861. $x''' + x'' - 2x = 5e^t, \quad x(0) = 0, \quad x'(0) = 1, \quad x''(0) = 2.$

862. $x'' + x = 8\sqrt{2}\sin\left(t + \frac{\pi}{4}\right), \quad x(0) = 0, \quad x'(0) = -4.$

863. $x'' + 4x = 2\cos^2 t, \quad x(0) = 0, \quad x'(0) = 0.$

24.3. Solving systems of linear differential equations with constant coefficients. Suppose it is required to find the solution of a system of two equations with constant coefficients

$$\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y + f_1(t), \\ \dfrac{dy}{dt} = a_2x + b_2y + f_2(t) \end{cases} \tag{23}$$

satisfying the initial conditions

$$x(0) = x_0, \quad y(0) = y_0. \tag{24}$$

We shall assume that the functions $f_1(t)$, $f_2(t)$, $x(t)$, $y(t)$, as well as $x'(t)$ and $y'(t)$ are original-functions. Let

$$x(t) \doteqdot X(p), \quad y(t) \doteqdot Y(p),$$
$$f_1(t) \doteqdot F_1(p), \quad f_2(t) \doteqdot F_2(p).$$

By the rule of differentiating originals and taking into account (24) we have

$$x'(t) \doteqdot pX(p) - x_0, \quad y'(t) \doteqdot pY(p) - y_0.$$

Applying the Laplace transformation to both sides of each equation of system (23) we obtain the operator system

$$\begin{cases} pX(p) = a_1X(p) + b_1Y(p) + F_1(p) + x_0, \\ pY(p) = a_2X(p) + b_2Y(p) + F_2(p) + y_0. \end{cases}$$

It is a linear algebraic system of two equations in two unknowns $X(p)$ and $Y(p)$. Solving the system we find $X(p)$ and $Y(p)$ and then, going over to the originals, obtain the solution $x(t)$, $y(t)$ of system (23) satisfying the initial conditions (24). One can similarly solve linear systems of the form

$$\frac{dx_k}{dt} = \sum_{l=1}^{n} a_{kl}x_l + f_k(t), \qquad a_{kl} = \text{const},$$
$$x_k(0) = x_k^0, \quad k = 1, 2, \ldots, n.$$

Example 6. Find the solution of the system

$$\begin{cases} \dfrac{dx}{dt} = -7x + y + 5, \\ \dfrac{dy}{dt} = -2x - 5y - 37t \end{cases}$$

satisfying the initial conditions $x(0) = 0$, $y(0) = 0$.

Solution. Since $5 \risingdotseq 5/p$, $-37t = -37/p^2$ and $x_0 = y_0 = 0$, the operator system is of the form

$$\begin{cases} pX(p) = -7X(p) + Y(p) + \dfrac{5}{p}, \\ pY(p) = -2X(p) - 5Y(p) - \dfrac{37}{p^2}. \end{cases}$$

Solving it we get

$$X(p) = \frac{5p^2 + 25p - 37}{p^2(p^2 + 12p + 37)}, \quad Y(p) = \frac{-47p - 259}{p^2(p^2 + 12p + 37)}.$$

We expand the fractions of the right-hand sides into partial fractions:

$$X(p) = \frac{1}{p} - \frac{1}{p^2} - \frac{p+6}{p^2 + 12p + 37},$$

$$y(p) = \frac{1}{p} - \frac{7}{p^2} - \frac{p+5}{p^2 + 12p + 37},$$

or

$$X(p) = \frac{1}{p} - \frac{1}{p^2} - \frac{p+6}{(p+6)^2 + 1},$$

$$Y(p) = \frac{1}{p} - \frac{7}{p^2} - \frac{p+6}{(p+6)^2 + 1} + \frac{1}{(p+6)^2 + 1}.$$

Going over to the originals we obtain the desired solution

$$x(t) = 1 - t - e^{-6t} \cos t, \quad y(t) = 1 - 7t + e^{-6t} \cos t + e^{-6t} \sin t.$$

In the problems below solve the systems of equations by the operator method:

864. $\begin{cases} \dfrac{dx}{dt} + y = 0, \\ \dfrac{dy}{dt} + x = 0, \end{cases} \quad x(0) = 2, \quad y(0) = 0.$

865. $\begin{cases} \dfrac{dx}{dt}+x-2y=0, \\ \dfrac{dy}{dt}+x+4y=0, \end{cases} \quad x(0)=y(0)=1.$

866. $\begin{cases} \dfrac{dx}{dt}=-y, \\ \dfrac{dy}{dt}=2(x+y), \end{cases} \quad x(0)=y(0)=1.$

867. $\begin{cases} \dfrac{dx}{dt}+2y=3t, \\ \dfrac{dy}{dt}-2x=4, \end{cases} \quad x(0)=2, \quad y(0)=3.$

868. $\begin{cases} \dfrac{dx}{dt}+x=y+e^{t}, \\ \dfrac{dy}{dt}+y=x+e^{t}, \end{cases} \quad x(0)=y(0)=1.$

869. $\begin{cases} \dfrac{dx}{dt}-\dfrac{dy}{dt}=-\sin t, \\ \dfrac{dx}{dt}+\dfrac{dy}{dt}=\cos t, \end{cases} \quad x(0)=\dfrac{1}{2}, \quad y(0)=-\dfrac{1}{2}.$

870. $\begin{cases} \dfrac{dx}{dt}=y-z, \\ \dfrac{dy}{dt}=x+y, \\ \dfrac{dz}{dt}=x+z, \end{cases} \quad x(0)=1, \quad y(0)=2, \quad z(0)=3.$

871. $\begin{cases} \dfrac{dx}{dt}=4y+z, \\ \dfrac{dy}{dt}=z, \\ \dfrac{dz}{dt}=4y, \end{cases} \quad x(0)=5, \quad y(0)=0, \quad z(0)=4.$

872. $\begin{cases} \dfrac{dx}{dt}+2\dfrac{dy}{dt}+x+y+z=0, \\ \dfrac{dx}{dt}+\dfrac{dy}{dt}+x+z=0, \\ \dfrac{dz}{dt}-2\dfrac{dy}{dt}-y=0, \end{cases} \quad x(0)=y(0)=1, \quad z(0)=-2.$

873. $\begin{cases} \dfrac{dx}{dt}-\dfrac{dy}{dt}-2x+2y=1-2t, \\ \dfrac{d^2x}{dt^2}+2\dfrac{dy}{dt}+x=0, \end{cases}$ $x(0)=y(0)=x'(0)=0.$

874. $\begin{cases} \dfrac{d^2x}{dt^2}=y, \\ \dfrac{d^2y}{dt^2}=x, \end{cases}$ $x(0)=y(0)=1,\ x'(0)=2,\ y'(0)=0.$

875. $\begin{cases} \dfrac{d^2x}{dt^2}=x-4y, & x(0)=2, \quad y(0)=0, \\ \dfrac{d^2y}{dt^2}=-x+y, & x'(0)=-\sqrt{3}, \quad y'(0)=\dfrac{\sqrt{3}}{2}. \end{cases}$

876. $\begin{cases} \dfrac{d^2x}{dt^2}+\dfrac{dy}{dt}=e^t-x, \\ \dfrac{d^2y}{dt^2}+\dfrac{dx}{dt}=1, \end{cases}$ $x(0)=1, \quad y(0)=0, \quad x'(0)=2, \quad y'(0)=-1.$

877. $\begin{cases} \dfrac{d^2x}{dt^2}+x+y=5, \\ \dfrac{d^2y}{dt^2}-4x-3y=-3, \end{cases}$ $x(0)=y(0)=x'(0)=y'(0)=0.$

878. $\begin{cases} \dfrac{dx}{dt}+4y+2x=4t+1, \\ \dfrac{dy}{dt}+x-y=\dfrac{3}{2}t^2, \end{cases}$ $x(0)=y(0)=0.$

879. $\begin{cases} \dfrac{dx}{dt}+y-2x=0, \\ \dfrac{dy}{dt}+x-2y=-5e^t\sin t, \end{cases}$ $x(0)=2, \quad y(0)=3.$

CHAPTER IV

STABILITY THEORY

25. Lyapunov stability. Basic concepts and definitions

Let the following system of differential equations be given

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n, t), \quad i = 1, 2, \ldots, n. \tag{1}$$

A solution $\varphi_i(t)$, $i = 1, 2, \ldots, n$, of system (1) satisfying the initial conditions $\varphi_i(t_0) = \varphi_{i0}$, $i = 1, 2, \ldots, n$, is said to be *a Lyapunov stable solution* if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for *each* solution $x_i(t)$, $i = 1, 2, \ldots, n$, of system (1) whose initial values satisfy the conditions

$$|x_i(t_0) - \varphi_{i0}| < \delta, \quad i = 1, 2, \ldots, n \tag{2}$$

the inequalities

$$|x_i(t) - \varphi_i(t)| < \varepsilon, \quad i = 1, 2, \ldots, n \tag{3}$$

hold for all $t \geqslant t_0$.

If for an arbitrarily small $\delta > 0$ inequalities (3) fail to hold for at least one solution $x_i(t)$, $i = 1, 2, \ldots, n$, then the solution $\varphi_i(t)$ is said to be *unstable*.

If under condition (2) besides inequalities (3) the condition

$$\lim_{t \to \infty} |x_i(t) - \varphi_i(t)| = 0, \quad i = 1, 2, \ldots, n \tag{4}$$

also holds, then the solution $\varphi_i(t)$, $i = 1, 2, \ldots, n$, is said to be *asymptotically stable*.

Investigating a solution $\varphi_i(t)$, $i = 1, 2, \ldots, n$, of system (1) for stability can be reduced to investigating for stability the zero (trivial) solution $x_i \equiv 0$, $i = 1, 2, \ldots, n$, of some system similar to system (1),

$$\frac{dx_i}{dt} = F_i(x_1, x_2, \ldots, x_n, t), \quad i = 1, 2, \ldots, n, \tag{1'}$$

where $F_i(0, 0, \ldots, 0, t) \equiv 0$, $i = 1, 2, \ldots, n$.

A point $x_i = 0$, $i = 1, 2, \ldots, n$, is said to be *a stationary point* of system (1′).

As applied to the stationary point the definitions of stability and instability can be formulated as follows. *A stationary point* $x_i = 0$, $i = 1, 2, \ldots, n$, *is stable according to Lyapunov* if whatever $\varepsilon > 0$ there exists $\delta > 0$ such that

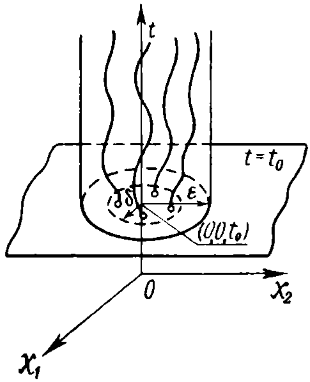

Fig. 30

for any solution $x_i(t)$, $i = 1, 2, \ldots, n$, whose initial data $x_{i0} = x_i(t_0)$, $i = 1, 2, \ldots, n$, satisfy the condition

$$|x_{i0}| < \delta, \quad i = 1, 2, \ldots, n, \tag{2'}$$

the inequalities

$$|x_i(t)| < \varepsilon, \quad i = 1, 2, \ldots, n \tag{3'}$$

hold for all $t \geqslant t_0$.

Geometrically, for the case $n = 2$ this implies the following. However narrow a cylinder of radius ε with the Ot axis may be, there is a δ-neighbourhood of the point $(0, 0, t_0)$ in the plane $t = t_0$ such that all integral curves

$$x_1 = x_1(t), \quad x_2 = x_2(t)$$

emanating from that neighbourhood will remain inside the cylinder for all $t \geqslant t_0$ (Fig. 30).

If besides inequalities (3) the condition $\lim\limits_{t \to +\infty} |x_i(t)| = 0$, $i = 1, 2, \ldots, n$, also holds, then *the stability is asymptotic.*

A stationary point $x_i = 0$, $i = 1, 2, \ldots, n$, is unstable if for an arbitrarily small $\delta > 0$ condition (3') does not hold for at least one solution $x_i(t)$, $i = 1, 2, \ldots, n$.

Example 1. Proceeding from the definition of Lyapunov stability, investigate for stability the solution of the equation

$$\frac{dx}{dt} = 1 + t - x \tag{5}$$

satisfying the initial condition

$$x(0) = 0.$$

Solution. Equation (5) is a nonhomogeneous linear equation. Its general solution is $x(t) = Ce^{-t} + t$. The initial condition $x(0) = 0$ is satisfied by the solution

$$\varphi(t) = t \tag{6}$$

of equation (5). The initial condition $x(0) = x_0$ is satisfied by the solution

$$x(t) = x_0 e^{-t} + t. \tag{7}$$

We consider the difference of solutions (7) and (6) of equation (5) and write it as

$$x(t) - \varphi(t) = x_0 e^{-t} + t - t = (x_0 - 0)\, e^{-t}.$$

Hence it is seen that for any $\varepsilon > 0$ there exists $\delta > 0$ (for example, $\delta = \varepsilon$) such that for any solution $x(t)$ of equation (5) whose initial values satisfy the condition $|x_0 - 0| < \delta$ the inequality

$$|x(t) - \varphi(t)| = |x_0 - 0|\, e^{-t} < -\varepsilon$$

holds for all $t \geqslant 0$. Therefore the solution $\varphi(t) = t$ is stable. Moreover, since

$$\lim_{t \to +\infty} |x(t) - \varphi(t)| = \lim_{t \to +\infty} |x_0 - 0|\, e^{-t} = 0$$

the solution $\varphi(t) = t$ is asymptotically stable.

That solution $\varphi(t)$ is unbounded when $t \to +\infty$. ◆

The above example shows that the stability of the solution of a differential equation does not imply the boundedness of the solution.

Example 2. Consider the equation [4]:

$$\frac{dx}{dt} = \sin^2 x. \tag{8}$$

It has the obvious solutions

$$x = k\pi, \quad k = 0, \pm 1, \pm 2, \ldots . \tag{9}$$

We integrate equation (8):

$$\cot x = C - t \quad \text{or} \quad \cot x = \cot x_0 - t,$$

whence

$$x = \operatorname{arc\,cot} (\cot x_0 - t), \quad x \neq k\pi. \tag{10}$$

All solutions (9) and (10) are bounded in $(-\infty, +\infty)$. The solution $x(t) \equiv 0$ is, however, unstable when $t \to +\infty$,

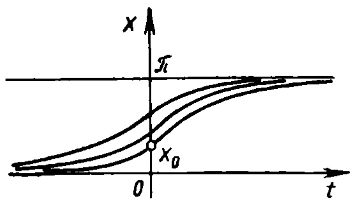

Fig. 31

since for any $x_0 \in (0, \pi)$ we have $\lim\limits_{t \to +\infty} x(t) = \pi$. Therefore, the boundedness of solutions of a differential equation does not imply their stability (Fig. 31).

This phenomenon is characteristic of nonlinear equations and systems.

Example 3. Proceeding from the definition of Lyapunov stability show that the solution of the system

$$\begin{cases} \dfrac{dx}{dt} = -y, \\ \dfrac{dy}{dt} = x \end{cases} \tag{11}$$

satisfying the initial conditions $x(0) = 0$, $y(0) = 0$ is stable.

Solution. The solution of system (11) satisfying the given initial conditions is $x(t) \equiv 0$, $y(t) \equiv 0$. Any solution of the system satisfying the conditions $x(0) = x_0$, $y(0) = y_0$ is of the form

$$x(t) = x_0 \cos t - y_0 \sin t, \quad y(t) = x_0 \sin t + y_0 \cos t.$$

We shall take an arbitrary $\varepsilon > 0$ and show that there exists $\delta(\varepsilon) > 0$ such that for $|x_0 - 0| < \delta$, $|y_0 - 0| < \delta$ the inequalities

$$|x(t) - 0| = |x_0 \cos t - y_0 \sin t| < \varepsilon,$$

$$|y(t) - 0| = |x_0 \sin t + y_0 \cos t| < \varepsilon$$

hold for all $t \geqslant 0$.

This exactly means according to the definition that the zero solution $x(t) \equiv 0$, $y(t) \equiv 0$ of system (11) is a Lyapunov stable solution. Obviously we have

$$|x_0 \cos t - y_0 \sin t| \leqslant |x_0 \cos t| + |y_0 \sin t| \leqslant |x_0| + |y_0|,$$

$$|x_0 \sin t + y_0 \cos t| \leqslant |x_0 \sin t| + |y_0 \cos t| \leqslant |x_0| + |y_0| \qquad (12)$$

for all t. Therefore, if $|x_0| + |y_0| < \varepsilon$, then so much the more

$$|x_0 \cos t - y_0 \sin t| < \varepsilon,$$

$$|x_0 \sin t + y_0 \cos t| < \varepsilon \qquad (13)$$

for all t.

Consequently, if we take, for example, $\delta(\varepsilon) = \varepsilon/2$, then by (12) inequalities (13) will hold for all $t \geqslant 0$ when $|x_0| < \delta$ and $|y_0| < \delta$, i.e. the zero solution of system (11) is indeed a Lyapunov stable solution, but its stability is not asymptotic.

Theorem. *Solutions of a system of linear differential equations*

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)\, x_j + f_i(t), \quad i = 1, 2, \ldots, n$$

are all either simultaneously stable or unstable.

This is not true of nonlinear systems some solutions of which may be stable and some unstable.

Example 4. Consider the nonlinear equation

$$\frac{dx}{dt} = 1 - x^2(t). \qquad (14)$$

It has obvious solutions $\varphi(t) = -1$ and $\varphi(t) = 1$.

The solution $\varphi(t) = -1$ of this equation is unstable and the solution $\varphi(t) = 1$ is asymptotically stable. Indeed,

for $t \to +\infty$ all solutions of equation (14)

$$x(t) = \frac{(1+x_0)\, e^{2(t-t_0)} - (1-x_0)}{(1+x_0)\, e^{2(t-t_0)} + (1-x_0)} \quad (x_0 \neq -1)$$

tend to $+1$. It means according to the definition that the solution $\varphi(t) \equiv 1$ of the equation is asymptotically stable.

Proceeding from the definition of Lyapunov stability investigate the following equations and systems of equations for stability:

880. $\dfrac{dx}{dt} = x + t, \quad x(0) = 1.$

881. $\dfrac{dx}{dt} = 2t(x+1), \quad x(0) = 0.$

882. $\dfrac{dx}{dt} = -x + t^2, \quad x(1) = 1.$

883. $\dfrac{dx}{dt} = 2 + t, \quad x(0) = 1.$

884. $\begin{cases} \dfrac{dx}{dt} = x - 13y, \\ \dfrac{dy}{dt} = \dfrac{1}{4}x - 2y, \end{cases} \quad x(0) = y(0) = 0.$

885. $\begin{cases} \dfrac{dx}{dt} = -x - 9y, \\ \dfrac{dy}{dt} = x - y. \end{cases} \quad x(0) = y(0) = 0.$

26. The simplest types of stationary points

Consider a system of two homogeneous linear differential equations with constant coefficients

$$\begin{cases} \dfrac{dx}{dt} = a_{11}x + a_{12}y, \\ \dfrac{dy}{dt} = a_{21}x + a_{22}y \end{cases} \tag{1}$$

with

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0.$$

A point $x = 0$, $y = 0$ in which the right-hand sides of the equations of system (1) vanish is called *a stationary point of system* (1).

In order for a stationary point of system (1) to be investigated it is necessary to set up the characteristic equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0 \qquad (2)$$

and find its roots λ_1 and λ_2.

The following cases are possible:

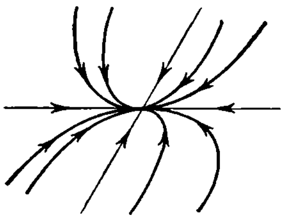

Fig. 32

1. The roots λ_1, λ_2 of the characteristic equation (2) are real and distinct:

(a) $\lambda_1 < 0$, $\lambda_2 < 0$. The stationary point is asymptotically stable (a stable node, Fig. 32);

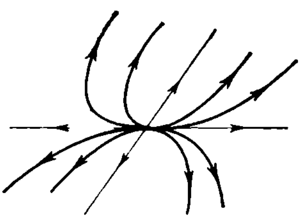

Fig. 33

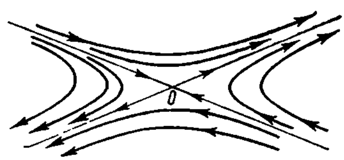

Fig. 34

(b) $\lambda_1 > 0$, $\lambda_2 > 0$. The stationary point is unstable (an unstable node, Fig. 33);

(c) $\lambda_1 > 0$, $\lambda_2 < 0$. The stationary point is unstable (a saddle point, Fig. 34).

2. The roots of the characteristic equation (2) are complex: $\lambda_1 = p + iq$, $\lambda_2 = p - iq$:

(a) $p < 0$, $q \neq 0$. The stationary point is asymptotically stable (a stable focus, Fig. 35);

(b) $p > 0$, $q \neq 0$. The stationary point is unstable (an unstable focus, Fig. 36);

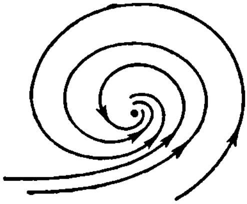

Fig. 35

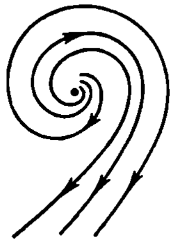

Fig. 36

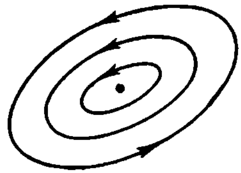

Fig. 37

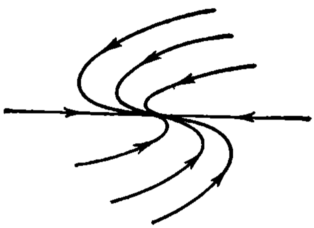

Fig. 38

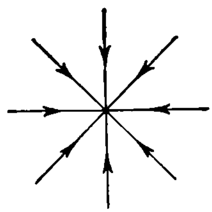

Fig. 39

(c) $p = 0$, $q \neq 0$. The stationary point is stable (a mid-point, Fig. 37).

3. The roots $\lambda_1 = \lambda_2$ are multiple:

(a) $\lambda_1 = \lambda_2 < 0$. The stationary point is asymptotically stable (a stable node, Figs. 38, 39);

(b) $\lambda_1 = \lambda_2 > 0$. The stationary point is unstable (an unstable node, Figs. 40, 41).

Example 1. Determine the character of the stationary point (0, 0) of the system

$$\begin{cases} \dfrac{dx}{dt} = 5x - y, \\ \dfrac{dy}{dt} = 2x + y. \end{cases}$$

Solution. We set up the characteristic equation

$$\begin{vmatrix} 5-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - 6\lambda + 7 = 0.$$

Its roots $\lambda_1 = 3 + \sqrt{2} > 0$, $\lambda_2 = 3 - \sqrt{2} > 0$ are real, distinct, and positive. Therefore the stationary point (0, 0) is an unstable node.

It is possible to visualize the relation between the types of stationary points and the values of the roots of the char-

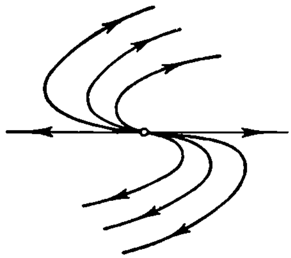

Fig. 40

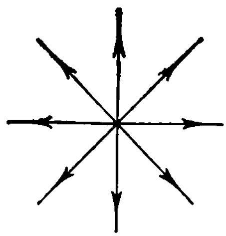

Fig. 41

acteristic equation (2). To do this we introduce the notation $\sigma = -(a_{11} + a_{22})$, $\Delta = a_{11}a_{22} - a_{12}a_{21}$. Then the characteristic equation will be written as $\lambda^2 + \sigma\lambda + \Delta = 0$.

Consider a plane with rectangular coordinates Δ and σ and mark on it regions corresponding to different types of stationary points (Fig. 42). It follows from the above classification that conditions for the stability of a stationary point are $\operatorname{Re}\lambda_1 < 0$, $\operatorname{Re}\lambda_2 < 0$. They hold for $\Delta > 0$ and $\sigma > 0$, i.e. for the points of the first quarter.

If λ_1 and λ_2 are complex, then the stationary point is of the focus type. This condition is satisfied by the points which lie between the branches of the parabola $\sigma^2 = 4\Delta$ and do not belong to the axis $O\Delta$ ($\sigma^2 < 4\Delta$, $\sigma \neq 0$).

The points of the semi-axis $\sigma = 0$ for which $\Delta > 0$ correspond to stationary points of the midpoint type.

The points lying outside the parabola $\sigma^2 = 4\Delta$ ($\sigma^2 > 4\Delta$) correspond to stationary points of the node type.

The region $O\Delta\sigma$, where $\Delta < 0$, contains stationary points of the saddle-point type.

Excluding singular cases (passage through the origin of coordinates), it is obvious that a saddle point may become

a node, stable or unstable (Fig. 42). A stable node may become either a saddle point or a stable focus. The case of equal roots $\lambda_1 = \lambda_2$ corresponds to the boundary between the nodes and foci, i.e. to the parabola $\sigma^2 = 4\Delta$.

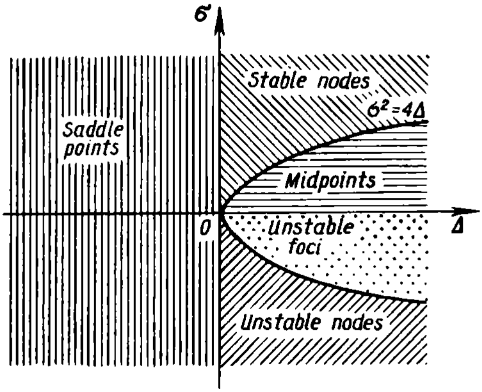

Fig. 42

Example 2. Investigate the equation of elastic vibrations

$$\frac{d^2x}{dt^2} + 2\alpha \frac{dx}{dt} + \beta^2 x = 0 \tag{3}$$

taking into account the friction and resistance of the medium (for $\alpha > 0$).

Solution. We go over from equation (3) to an equivalent system of equations

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -2\alpha y - \beta^2 x. \end{cases} \tag{4}$$

In order to determine the character of the stationary point (0, 0) of system (4) we set up the characteristic equation

$$\begin{vmatrix} -\lambda & 1 \\ -\beta^2 & -2\alpha - \lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 + 2\alpha\lambda + \beta^2 = 0;$$

hence

$$\lambda_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \beta^2}. \tag{5}$$

We consider the following cases: (a) $\alpha = 0$ (the medium has no resistance). From (5) we get $\lambda_{1,2} = \pm i\beta$. The stationary point is stable, a midpoint (all motions are periodic);

(b) $\alpha > 0$, $\alpha^2 - \beta^2 < 0$. The roots λ_1 and λ_2 are complex conjugate, with $\operatorname{Re}\lambda < 0$. The stationary point is a stable focus (damped oscillation);

(c) $\alpha < 0$ (the case of "negative friction"), $\alpha^2 - \beta^2 < 0$. The roots λ_1 and λ_2 are complex conjugate, with $\operatorname{Re}\lambda > 0$. The stationary point is an unstable focus;

(d) $\alpha > 0$, $\alpha^2 - \beta^2 \geqslant 0$ (the resistance of the medium is great $\alpha \geqslant \beta$). The roots λ_1 and λ_2 are real and negative. The stationary point is a stable node (all solutions are damped and nonoscillatory);

(e) $\alpha < 0$, $\alpha^2 - \beta^2 \geqslant 0$ (the case of large "negative friction"). The roots λ_1 and λ_2 are real and positive. The stationary point is an unstable node.

Determine the character of the stationary points for the following systems of differential equations:

886. $\begin{cases} \dfrac{dx}{dt} = 3x + y, \\ \dfrac{dy}{dt} = -2x + y. \end{cases}$

887. $\begin{cases} \dfrac{dx}{dt} = -x + 2y, \\ \dfrac{dy}{dt} = x + y. \end{cases}$

888. $\begin{cases} \dfrac{dx}{dt} = -x + 3y, \\ \dfrac{dy}{dt} = -x + y. \end{cases}$

889. $\begin{cases} \dfrac{dx}{dt} = -2x - y, \\ \dfrac{dy}{dt} = 3x - y. \end{cases}$

890. $\begin{cases} \dfrac{dx}{dt} = -2x + \dfrac{5}{7}y, \\ \dfrac{dy}{dt} = 7x - 3y. \end{cases}$

891. $\begin{cases} \dfrac{dx}{dt} = 3x - y, \\ \dfrac{dy}{dt} = x + y. \end{cases}$

892. $\begin{cases} \dfrac{dx}{dt} = 3x, \\ \dfrac{dy}{dt} = 3y. \end{cases}$

893. For what values of α is the stationary point (0, 0) of the system

$$\begin{cases} \dfrac{dx}{dt} = -3x + \alpha y, \\ \dfrac{dy}{dt} = 2x + y \end{cases}$$

stable?

Consider the system of homogeneous linear differential equations with constant coefficients

$$\frac{dx_i}{dt}=\sum_{j=1}^{n} a_{ij}x_j, \quad i=1,\ 2,\ \ldots,\ n\ (n\geqslant 2). \tag{6}$$

It has similar types of location of integral curves about the origin of coordinates (*the generalized saddle point*, *generalized node*, etc.).

Theorem. *If all roots of the characteristic equation for system* (6) *have a negative real part, then the stationary point of system* (6) $x_i = 0$, $i = 1, 2, \ldots, n$, *is asymptotically stable. If at least one root of the characteristic equation has a positive real part, then the stationary point is unstable.*

Example 3. Is the stationary point (0, 0, 0) of the system

$$\begin{cases} \dfrac{dx}{dt}=-x+z, \\ \dfrac{dy}{dt}=-2y-z, \\ \dfrac{dz}{dt}=y-z \end{cases}$$

stable?

Solution. We set up the characteristic equation

$$\begin{vmatrix} -1-\lambda & 0 & 1 \\ 0 & -2-\lambda & -1 \\ 0 & 1 & -1-\lambda \end{vmatrix}=0$$

or $(1 + \lambda)(\lambda^2 + 3\lambda + 3) = 0$. The roots of this equation $\lambda_1 = -1$, $\lambda_{2,3} = -\frac{3}{2} \pm i\frac{\sqrt{3}}{2}$ have negative real parts. Therefore the stationary point of the given system is asymptotically stable.

Investigate the stationary point O (0, 0, 0) of the following systems for stability:

894. (a) $\begin{cases} \dfrac{dx}{dt}=-x+y+5z, \\ \dfrac{dy}{dt}=-2y+z, \\ \dfrac{dz}{dt}=-3z; \end{cases}$ (b) $\begin{cases} \dfrac{dx}{dt}=x, \\ \dfrac{dy}{dt}=2x-y, \\ \dfrac{dz}{dt}=x+y-z; \end{cases}$

$$
\text{(c)}\quad \begin{cases} \dfrac{dx}{dt} = -2x - y, \\ \dfrac{dy}{dt} = x - 2y, \\ \dfrac{dz}{dt} = x + 3y - z. \end{cases}
$$

27. The method of Lyapunov functions

The method of Lyapunov functions is to investigate directly the stability of the equilibrium position of the system

$$
\frac{dx_i}{dt} = f_i(t, x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n
$$

with the help of a suitably selected function $V(t, x_1, \ldots, x_n)$, *the Lyapunov function*, this being done without finding beforehand any solutions of the system.

We restrict ourselves to the consideration of autonomous systems

$$
\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n \tag{1}
$$

for which $x_i \equiv 0$, $i = 1, 2, \ldots, n$, is a stationary point.

The function $V(x_1, x_2, \ldots, x_n)$ defined in some neighbourhood of the origin of coordinates is said *to be of fixed sign* (positive definite or negative definite) if in the domain

$$
|x_i| \leqslant h, \quad i = 1, 2, \ldots, n, \tag{2}
$$

h being a sufficiently small positive number, it can take values of only one definite sign and vanishes only when $x_1 = x_2 = \ldots = x_n = 0$. Thus in the case $n = 3$ the functions

$$
V = x_1^2 + x_2^2 + x_3^2 \quad \text{and} \quad V = x_1^2 + 2x_1x_2 + 2x_2^2 + x_3^2
$$

are positive definite, it being possible here to take the quantity $h > 0$ arbitrarily large.

The function $V(x_1, x_2, \ldots, x_n)$ is said *to be of constant signs* (positive or negative) if in domain (2) it can take values of only one definite sign but can also vanish when $x_1^2 + x_2^2 + \ldots + x_n^2 \neq 0$. For example, the function

$$
V(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_1x_2 + x_3^2
$$

is of constant sign (positive). Indeed, the function $V(x_1, x_2, x_3)$ may be written thus: $V(x_1, x_2, x_3) = (x_1 + x_2)^2 + x_3^2$, whence we see that it vanishes also when $x_1^2 + x_2^2 + x_3^2 \neq 0$, namely when $x_3 = 0$ and for any x_1 and x_2 such that $x_1 = -x_2$.

Let $V(x_1, x_2, \ldots, x_n)$ be a differentiable function of its variables and let $x_1, x_2, \ldots, x_n$ be some functions of time satisfying the system of differential equations (1). Then for the total derivative of the function V with respect to time we have:

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \cdot \frac{dx_i}{dt} = \sum_{i=1}^{n} \frac{\partial V}{x_i} f_i(x_1, x_2, \ldots, x_n). \tag{3}$$

The quantity $\frac{dV}{dt}$ defined by formula (3) is called *the total derivative of the function V with respect to time* composed by virtue of the system of equations (1).

Theorem 1 (Lyapunov's stability theorem). *If for a system of differential equations* (1) *there exists a function of fixed sign $V(x_1, x_2, \ldots, x_n)$ (a Lyapunov function) whose total derivative $\frac{dV}{dt}$ with respect to time composed by virtue of system* (1) *is a function of constant signs, of sign opposite to that of V, or identically equal to zero, then the stationary point $x_i = 0$, $i = 1, 2, \ldots, n$, of system* (1) *is stable.*

Theorem 2 (Lyapunov's asymptotic-stability theorem). *If for a system of differential equations* (1) *there exists a function of fixed sign $V(x_1, x_2, \ldots, x_n)$ whose total derivative with respect to time composed by virtue of system* (1) *is also a function of fixed sign, of sign opposite to that of V, then the stationary point $x_i = 0$ of system* (1) *is asymptotically stable.*

Example 1. Consider the system

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -x. \end{cases} \tag{4}$$

We choose the function $V = x^2 + y^2$ as the function $V(x, y)$. It is positive definite. The derivative of the function V is by virtue of system (4) equal to

$$\frac{dV}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2xy - 2xy \equiv 0.$$

It follows from Theorem 1 that the stationary point O (0, 0) of system (4) is stable. It is not asymptotically stable, however: the trajectories of system (4) are circles and they do not tend to the point O (0, 0) as $t \to +\infty$.

Example 2. Consider the system

$$\begin{cases} \frac{dx}{dt} = y - x^3, \\ \frac{dy}{dt} = -x - 3y^3. \end{cases} \tag{5}$$

Taking again $V(x, y) = x^2 + y^2$ we find that

$$\frac{dV}{dt} = 2x(y - x^3) + 2y(-x - 3y^3) = -2(x^4 + 3y^4).$$

Thus $\frac{dV}{dt}$ is a negative definite function. By Theorem 2 the stationary point O (0, 0) of system (5) is asymptotically stable.

There is no general method for constructing Lyapunov functions. In the simplest cases a Lyapunov function may be sought in the form

$$V(x, y) = ax^2 + by^2, \quad V(x, y) = ax^4 + by^4,$$
$$V(x, y) = ax^4 + by^2 \ (a > 0, \ b > 0), \text{ etc.}$$

Example 3. Using a Lyapunov function investigate for stability the trivial solution $x \equiv 0$, $y \equiv 0$ of the system

$$\begin{cases} \frac{dx}{dt} = -x - 2y + x^2y^2, \\ \frac{dy}{dt} = x - \frac{y}{2} - \frac{x^3y}{2}. \end{cases}$$

Solution. We shall seek a Lyapunov function in the form $V = ax^2 + by^2$, where $a > 0$, $b > 0$ are arbitrary parameters. We have

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = 2ax(-x - 2y + x^2y^2)$$
$$+ 2by\left(x - \frac{1}{2}y - \frac{1}{2}x^3y\right) = -(2ax^2 + by^2)$$
$$+ (2xy - x^3y^2)(b - 2a).$$

Setting $b = 2a$ we find that $\frac{dV}{dt} = -2a(x^2 + y^2) \leqslant 0$. Thus for any $a > 0$ and $b = 2a$ the function $V = ax^2 + 2ay^2$

is positive definite and its derivative $\frac{dV}{dt}$ composed by virtue of the given system is negative definite. It follows from Lyapunov's Theorem 2 that the trivial solution $x \equiv 0$, $y \equiv 0$ of the given system is asymptotically stable.

If we had not succeeded in finding the function $V(x, y)$ in the form indicated above, we should have sought it in the form $V = ax^4 + by^4$ or $V = ax^4 + by^2$, etc.

Theorem 3 (Lyapunov's instability theorem). *Let there exist for the system of differential equations* (1) *a function*

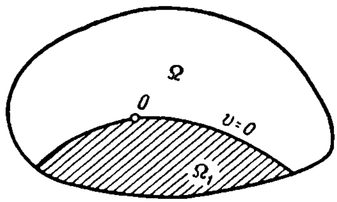

Fig. 43

differentiable in the neighbourhood of the origin of coordinates, $V(x_1, x_2, \ldots, x_n)$, *such that* $V(0, 0, \ldots, 0) = 0$. *If its total derivative* $\frac{dV}{dt}$ *composed by virtue of system* (1) *is a positive definite function and arbitrarily close to the origin of coordinates there are points in which the function* $V(x_1, x_2, \ldots, x_n)$ *takes positive values, then the stationary point* $x_i = 0$, $i = 1, 2, \ldots, n$, *is unstable.*

Theorem 4 (Chetayev's instability theorem). *Let for the system of differential equations* (1) *there exist a function* $v(x_1, x_2, \ldots, x_n)$ *continuously differentiable in some neighbourhood of a stationary point* $x_i = 0$, $i = 1, 2, \ldots, n$, *satisfying the following conditions in some closed neighbourhood of the stationary point:* (1) *in an arbitrarily small neighbourhood* Ω *of the stationary point* $x_i = 0$, $i = 1, 2, \ldots, n$ *there exists a domain* Ω_1 *in which* $v(x_1, x_2, \ldots, x_n) > 0$, *with* $v = 0$ *in the boundary points of* Ω_1 *that are interior for* Ω (Fig. 43).

(2) the stationary point $O(0, 0, \ldots, 0)$ *is a boundary point of the domain* Ω_1;

(3) *the derivative* $\frac{dv}{dt}$ *composed by virtue of system* (1) *is positive definite in the domain* Ω_1.

Then the stationary point $x_i = 0$, $i = 1, 2, \ldots, n$, of system (1) *is unstable.*

Example 4. Investigate the stationary point $x = 0$, $y = 0$ of the system

$$\begin{cases} \frac{dx}{dt} = x, \\ \frac{dy}{dt} = -y \end{cases}$$

for stability.

Solution. Take the function $v(x, y) = x^2 - y^2$. Then

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt} = 2x^2 + 2y^2$$

is a positive definite function. Since arbitrarily close to the origin of coordinates there are points in which $v > 0$ (for example, $v = x^2 > 0$ along the straight line $y = 0$), all the conditions of Theorem 3 hold and the stationary point $O(0, 0)$ is unstable (a saddle point).

Example 5. Investigate the stationary point $x = 0$, $y = 0$ of the system

$$\begin{cases} \frac{dx}{dt} = y^3 + x^5, \\ \frac{dy}{dt} = x^3 + y^5. \end{cases}$$

for stability.

Solution. The function $v = x^4 - y^4$ satisfies the conditions of Chetayev's theorem:

(1) $v > 0$ when $|x| > |y|$;

(2) $\frac{dv}{dt} = 4(x^8 - y^8)$ is positive definite in the domain $|x| > |y|$.

Hence the stationary point $x = 0$, $y = 0$ is unstable.

Investigate the trivial solutions of the systems below for stability:

895. $\begin{cases} \frac{dx}{dt} = -3y - 2x^3, \\ \frac{dy}{dt} = 2x - 3y^3. \end{cases}$

896. $\begin{cases} \frac{dx}{dt} = -xy^4, \\ \frac{dy}{dt} = x^4 y. \end{cases}$

897. $\begin{cases} \frac{dx}{dt} = x + 2xy^2, \\ \frac{dy}{dt} = -2y + 4x^2 y. \end{cases}$

898. $\begin{cases} \dfrac{dx}{dt} = -y - \dfrac{x}{2} - \dfrac{x^3}{4}, \\ \dfrac{dy}{dt} = x - \dfrac{y}{2} - \dfrac{1}{4} y^3. \end{cases}$

899. $\begin{cases} \dfrac{dx}{dt} = y + x^3, \\ \dfrac{dy}{dt} = -x + y^3. \end{cases}$

900. $\begin{cases} \dfrac{dx}{dt} = y + x^2y^2 - \dfrac{1}{4} x^5, \\ \dfrac{dy}{dt} = -2x - x^3y - \dfrac{1}{2} y^3. \end{cases}$

901. $\begin{cases} \dfrac{dx}{dt} = -x, \\ \dfrac{dy}{dt} = -y. \end{cases}$

902. $\begin{cases} \dfrac{dx}{dt} = x + x^3, \\ \dfrac{dy}{dt} = -y - y^3. \end{cases}$

903. $\begin{cases} \dfrac{dx}{dt} = xy^4 - 2x^3 - y, \\ \dfrac{dy}{dt} = 2x^2y^3 - y^7 + 2x. \end{cases}$

904. $\begin{cases} \dfrac{dx}{dt} = -2x - 3y, \\ \dfrac{dy}{dt} = x - y. \end{cases}$

905. $\begin{cases} \dfrac{dx}{dt} = x^5 + y^3, \\ \dfrac{dy}{dt} = x^3 - y^5. \end{cases}$

906. $\begin{cases} \dfrac{dx}{dt} = xy - x^3 + y, \\ \dfrac{dy}{dt} = x^4 - x^2y - x^3. \end{cases}$

907. $\begin{cases} \dfrac{dx}{dt} = x^3 + 2xy^2, \\ \dfrac{dy}{dt} = x^2y. \end{cases}$

908. $\begin{cases} \dfrac{dx}{dt} = -2y - x(x-y)^2, \\ \dfrac{dy}{dt} = 3x - \dfrac{3}{2} y(x-y)^2. \end{cases}$

909. Let $v = v(x_1, x_2, \ldots, x_n)$ be twice continuously differentiable positive definite function such that

$$v(0) = \frac{\partial v(0)}{\partial x_1} = \ldots = \frac{\partial v(0)}{\partial x_n} = 0.$$

Investigate the trivial solution $x_1 = 0, \ldots, x_n = 0$ of the system of differential equations below

$$\begin{cases} \dfrac{dx_1}{dt} = -\dfrac{\partial v}{\partial x_1}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dfrac{dx_n}{dt} = -\dfrac{\partial v}{\partial x_n} \end{cases}$$

for stability.

28. Stability in the first approximation

Let the following system of differential equations be given

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n \tag{1}$$

and let $x_i \equiv 0$, $i = 1, 2, \ldots, n$, be a stationary point of system (1), i.e. $f_i(0, 0, \ldots, 0) = 0$, $i = 1, 2, \ldots, n$. We shall assume that functions $f_i(x_1, x_2, \ldots, x_n)$ can be differentiated a sufficiently large number of times at the origin of coordinates.

We expand the functions f_i in the Taylor series of x in the neighbourhood of the origin of coordinates:

$$f_i(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} a_{ij}x_j + R_i(x_1, x_2, \ldots, x_n),$$

here $a_{ij} = \dfrac{\partial f_i(0, 0, \ldots, 0)}{\partial x_j}$ and R_i are terms of the second order of smallness with respect to $x_1, x_2, \ldots, x_n$.

The original system (1) will then be written as

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j + R_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n. \tag{2}$$

Instead of system (2) we shall consider the system

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j, \quad (i = 1, 2, \ldots, n) \; (a_{ij} + \text{const}) \tag{3}$$

called *the system of equations of the first approximation* for system (1).

The following propositions hold.

1. If all roots of the characteristic equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \dots & & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot \\ a_{n1} & a_{n2} \dots & & a_{nn}-\lambda \end{vmatrix} = 0 \tag{4}$$

have negative real parts, then zero solutions $x_i \equiv 0$, $i = 1, 2, \dots, n$, of system (3) and system (2) are asymptotically stable.

2. If at least one root of the characteristic equation (4) has a positive real part, then the zero solution of system (3) and system (2) is unstable.

It is said that investigation for stability in the first approximation is possible in cases 1 and 2.

In critical cases when the real parts of all roots of the characteristic equation (4) are nonpositive, with the real part of at least one root being zero, investigation for stability in the first approximation is in general impossible (nonlinear terms R_i starting to exert influence).

Example 1. Investigate the stationary point $x = 0$, $y = 0$ of the system

$$\begin{cases} \dot{x} = 2x + y - 5y^2, \\ \dot{y} = 3x + y + \dfrac{x^3}{2} \end{cases} \quad \left(\dot{x} = \frac{dx}{dt}, \quad \dot{y} = \frac{dy}{dt}\right) \tag{5}$$

for stability in the first approximation.

Solution. The system of the first approximation is

$$\begin{cases} \dot{x} = 2x + y, \\ \dot{y} = 3x + y; \end{cases} \tag{6}$$

the nonlinear terms satisfy the necessary conditions, their order being greater than or equal to two. We set up the characteristic equation for system (6)

$$\begin{vmatrix} 2-\lambda & 1 \\ 3 & 1-\lambda \end{vmatrix} = 0 \text{ or } \lambda^2 - 3\lambda - 1 = 0. \tag{7}$$

The roots of the characteristic equation (7) $\lambda_1 = \frac{3+\sqrt{13}}{3}$, $\lambda_2 = \frac{3-\sqrt{13}}{3}$ are real and $\lambda_1 > 0$. Therefore the zero solution $x=0$, $y=0$ of system (5) is unstable.

Example 2. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= y - x^3, \\ \frac{dy}{dt} &= -x - y^3. \end{aligned} \tag{8}$$

The stationary point $x \equiv 0$, $y \equiv 0$ of system (8) is asymptotically stable, since for this system the function $v = x^2 + y^2$ satisfies all the conditions of Lyapunov's asymptotic-stability theorem. In particular,

$$\frac{dv}{dt} = 2x(y - x^3) + 2y(-x - y^3) = -2(x^4 + y^4) \leqslant 0.$$

At the same time the stationary point $x \equiv 0$, $y \equiv 0$ of the system

$$\left\{ \begin{aligned} \frac{dx}{dt} &= y + x^3, \\ \frac{dy}{dt} &= -x + y^3 \end{aligned} \right. \tag{9}$$

is unstable by Chetayev's theorem; on taking $v = x^2 + y^2$ we have $\frac{dv}{dt} = 2(x^4 + y^4) \geqslant 0$.

Systems (8) and (9) have the same system of the first approximation

$$\left\{ \begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -x. \end{aligned} \right. \tag{10}$$

The characteristic equation for system (10)

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \text{ or } \lambda^2 + 1 = 0$$

has pure imaginary roots, so that the real parts of the roots of the characteristic equation are zero.

For the system of the first approximation (10) the origin of coordinates is a midpoint. Systems (8) and (9) are obtained

by means of a small perturbation of the right-hand sides of system (10) in the neighbourhood of the origin. However, these small perturbations result in closed trajectories turning into spirals approaching the origin and forming a stable focus in the point O (0, 0) (the case of (8)) or receding from

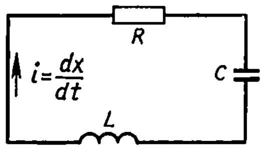

Fig. 44

the origin and forming an unstable focus in the point O (0, 0) (the case of (9)). Thus in the critical case nonlinear terms can influence the stability of a stationary point.

Example 3. Consider a closed circuit with nonlinear elements (Fig. 44); the equation of the circuit is

$$L\frac{d^2x}{dt^2}+R\frac{dx}{dt}+\frac{1}{C}x+g\left(x, \frac{dx}{dt}\right)=0. \tag{11}$$

Here x is the capacitor charge and hence $\frac{dx}{dt}$ is the current in the circuit; R is the resistance; L, the inductance; C, the capacitance; $g\left(x, \frac{dx}{dt}\right)$ are the nonlinear terms of order not less than 2, $g(0, 0)=0$. Equation (11) is equivalent to the system

$$\begin{cases} \frac{dx}{dt}=y, \\ \frac{dy}{dt}=-\frac{1}{LC}x-\frac{R}{L}y-\frac{1}{L}g(x, y), \end{cases} \tag{12}$$

for which the origin of coordinates O (0, 0) is a stationary point.

Consider the system of the first approximation

$$\begin{cases} \frac{dx}{dt}=y, \\ \frac{dy}{dt}=-\frac{1}{LC}x-\frac{R}{L}y. \end{cases} \tag{13}$$

The characteristic equation for system (13) is of the form

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{1}{LC} & -\frac{R}{L}-\lambda \end{vmatrix}=0 \quad \text{or} \quad \lambda^2+\frac{R\lambda}{L}+\frac{1}{LC}=0. \tag{14}$$

If $\frac{R^2}{L^2}<\frac{4}{LC}$, i.e. $R^2<\frac{4L}{C}$, then equation (14) has complex roots with a negative real part $p=-\frac{R}{4L}$ and so the origin of coordinates $O(0, 0)$ is asymptotically stable for systems (13) and (12).

If $R^2>\frac{4L}{C}$, then the origin of coordinates is also asymptotically stable (all the parameters R, L, C are positive).

The asymptotic stability of the stationary point is obvious from physical considerations; with a positive ohmic resistance the current inevitably disappears as t increases.

Investigate the zero solution $x=0$, $y=0$ of the following system for stability in the first approximation:

910. $\begin{cases} \dot{x}=x+2y-\sin y^2, \\ \dot{y}=-x-3y+x(e^{\frac{x^2}{2}}-1). \end{cases}$

911. $\begin{cases} \dot{x}=-x+3y+x^2\sin y, \\ \dot{y}=-x-4y+1-\cos y^2. \end{cases}$

912. $\begin{cases} \dot{x}=-2x+8\sin^2 y, \\ \dot{y}=x-3y+4x^3. \end{cases}$

913. $\begin{cases} \dot{x}=3x-22\sin y+x^2-y^3, \\ \dot{y}=\sin x-5y+e^{x^2}-1. \end{cases}$

914. $\begin{cases} \dot{x}=-10x+4e^{y}-4\cos y^2, \\ \dot{y}=2e^{x}-2-y+x^4. \end{cases}$

915. $\begin{cases} \dot{x}=7x+2\sin y-y^4, \\ \dot{y}=e^{x}-3y-1+\frac{5}{2}x^2. \end{cases}$

916. $\begin{cases} \dot{x} = -\frac{3}{2}x + \frac{1}{2}\sin 2y - x^3 y, \\ \dot{y} = -y - 2x + x^4 - y^7. \end{cases}$

917. $\begin{cases} \dot{x} = \frac{5}{2}xe^x - 3y + \sin x^2, \\ \dot{y} = 2x + ye^{-\frac{y^2}{2}} - y^4\cos x. \end{cases}$

918. $\begin{cases} \dot{x} = \frac{3}{4}\sin x - 7y(1-y)^{\frac{1}{3}} + x^3, \\ \dot{y} = \frac{2}{3}x - 3y\cos y - 11y^5. \end{cases}$

919. $\begin{cases} \dot{x} = \frac{1}{4}(e^x - 1) - 9y + x^4, \\ \dot{y} = \frac{1}{5}x - \sin y + y^{14}. \end{cases}$

920. $\begin{cases} \dot{x} = 5x + y\cos y - \frac{x^3}{3}, \\ \dot{y} = 3x + 2y + \frac{x^4}{12} - y^3 e^y. \end{cases}$

921. $\begin{cases} \dot{x} = 4y - x^3, \\ \dot{y} = -3x - y^3. \end{cases}$

922. $\begin{cases} \dot{x} = -2y - x^5, \\ \dot{y} = 2x - y^5. \end{cases}$

29. Stability of solutions of differential equations with respect to changes in the right-hand sides of the equations

Consider the differential equations

$$y' = f(x, y), \tag{1}$$

$$y' = f(x, y) + \Theta(x, y), \tag{2}$$

where the functions $f(x, y)$ and $\Theta(x, y)$ are continuous in a closed domain $\overline{G}$ of the xOy plane and the function $f(x, y)$

has a continuous partial derivative $\frac{\partial f}{\partial y}$ in that domain.

Let the inequality $|\Theta(x, y)| \leqslant \varepsilon$ hold in the domain $\bar{G}$. If $y = \varphi(x)$ and $y = \psi(x)$ are solutions of equations (1) and (2), respectively, satisfying the same initial condition $\varphi|_{x=x_0} = \psi|_{x=x_0} = y_0$, then

$$|\varphi(x) - \psi(x)| \leqslant \frac{\varepsilon}{M}\left(e^{M|x-x_0|} - 1\right), \tag{3}$$

where $M = \max\limits_{(x, y)\in\bar{G}} \left|\frac{\partial f}{\partial y}\right|$.

It is seen from estimate (3) that if the perturbation $\Theta(x, y)$ of the right-hand side of (1) is sufficiently small in the domain $\bar{G}$, then in the finite interval of x the difference in the absolute value of the solutions of equations (1) and (2) will be small. This allows one to solve approximately complicated differential equations by replacing them with reasonably chosen equations that are easier to solve. This last fact can be essentially used in solving differential equations connected with physical and engineering problems.

Example. In a square $Q\left\{-\frac{1}{2} \leqslant x \leqslant \frac{1}{2};\ -\frac{1}{2} \leqslant y \leqslant \frac{1}{2}\right\}$ find an approximate solution of the equation

$$y' = \sin(xy) \tag{4}$$

satisfying the initial condition

$$y|_{x=0} = 0.1 \tag{5}$$

and estimate the error.

Solution. We replace equation (4) by the equation

$$y' = xy, \tag{6}$$

$$y|_{x=0} = 0.1. \tag{7}$$

Under the initial condition (7) equation (6) has the solution $y = 0.1 \times e^{x^2/2}$ which remains inside the basic square Q for all $x \in [-1/2;\ 1/2]$.

By the existence and uniqueness theorem, under the initial condition (5) equation (4) has the unique solution $y = \psi(x)$ and as an approximate solution of problem (4), (5) we may take $y = 0.1 \times e^{x^2/2}$, the solution of problem (6), (7).

We estimate the difference

$$\Delta = |\varphi(x) - \psi(x)|, \quad -1/2 \leqslant x \leqslant 1/2,$$

where $\varphi(x) = 0.1 \times e^{x^2/2}$ is the solution of problem (6), (7). In the given case $f(x, y) = xy$ and $\left|\frac{\partial f}{\partial y}\right| = |x| \leqslant \frac{1}{2}$. We have $|\sin z - z| \leqslant \frac{|z|^3}{6}$, by the Taylor formula; therefore

$$|\sin xy - xy| \leqslant \frac{|xy|^3}{6} < \frac{1}{4^3 \times 6} = \frac{1}{384}$$

in the square Q.

We make use of estimate (3) by taking $\varepsilon = \frac{1}{384}$, $M = \max\limits_{(x, y) \in Q} \left|\frac{\partial f}{\partial y}\right| = \frac{1}{2}$:

$$\Delta = |\varphi(x) - \psi(x)| \leqslant \frac{1}{192}\left(e^{\frac{1}{2}|x|} - 1\right) < \frac{1}{650},$$

$$x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

It can easily be seen that the solution $\psi(x)$ of problem (4), (5) remains inside the basic square Q.

Estimate the difference in the value of the solutions of the given equations satisfying the same initial condition $y|_{x=x_0} = y_0$ in the given intervals ($x_0 = 0$):

923. $y' = \frac{y}{1+x} + x^2$,

$y' = \frac{y}{1+x} + x^2 + 0.01 \sin x$ in $[0.1]$.

924. $y' = e^{-\frac{\sin y}{1+x^2}}$,

$y' = e^{-\frac{\sin y}{1+x^2}} + \frac{\cos xy}{10(4+x^4)}$ in $[0.2]$.

925. $y' = \frac{1}{3} \arctan xy$,

$y' = \frac{1}{3} \arctan xy + 0.001e^{-x^2}$ in $[0.1]$.

30. The Routh-Hurwitz criterion

Consider a linear differential equation with constant real coefficients:

$$a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_ny = \\ = 0 \ (a_0, a_1, \ldots, a_n = \text{const}, \quad a_0 > 0). \quad (1)$$

The zero solution $y \equiv 0$ of equation (1) is asymptotically stable, if all the roots of the characteristic equation

$$f(\lambda) \equiv a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0 \quad (2)$$

have negative real parts.

The Routh-Hurwitz criterion. *For all roots of equation* (2) *to have negative real parts, it is necessary and sufficient that all the principal diagonal minors of the Hurwitz matrix*

$$\begin{pmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & \ldots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & a_n \end{pmatrix} \quad (3)$$

should be positive.

A Hurwitz matrix is composed as follows. The coefficients of polynomial (2), from a_1 to a_n, are written out in the main diagonal. The columns consist in turn of coefficients with only odd or only even subscripts, with the coefficient a_0 included in the latter. All the other entries of the matrix corresponding to coefficients with subscripts greater than n or less than 0 are set equal to zero. The principal diagonal minors of the Hurwitz matrix are of the form

$$\Delta_1 = a_1, \quad \Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \ \ldots$$

$$\Delta_n = \begin{vmatrix} a_1 & a_0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & \ldots & 0 \\ a_5 & a_4 & a_3 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & a_n \end{vmatrix}.$$

Thus the Hurwitz condition states: *for the solution* $y \equiv 0$ *of equation* (1) *to be stable it is necessary and sufficient that the relations*

$$\Delta_1 > 0, \quad \Delta_2 > 0, \ldots, \Delta_n > 0 \tag{4}$$

should hold.

Since $\Delta_n = a_n \Delta_{n-1}$, the condition $\Delta_n > 0$ may be replaced by the requirement $a_n > 0$.

Example. Investigate the zero solution of the equation

$$y^{\mathrm{IV}} + 5y''' + 13y'' + 19y' + 10y = 0 \tag{5}$$

for stability.

Solution. We set up the characteristic equation

$$f(\lambda) \equiv \lambda^4 + 5\lambda^3 + 13\lambda^2 + 19\lambda + 10 = 0.$$

Here $a_0 = 1$, $a_1 = 5$, $a_2 = 13$, $a_3 = 19$, $a_4 = 10$. We write out Hurwitz diagonal minors

$$\Delta_4 = \begin{vmatrix} 5 & 1 & 0 & 0 \\ 19 & 13 & 5 & 1 \\ 0 & 10 & 19 & 13 \\ 0 & 0 & 0 & 10 \end{vmatrix} = 4240 > 0,$$

$$\Delta_3 = \begin{vmatrix} 5 & 1 & 0 \\ 19 & 13 & 5 \\ 0 & 10 & 19 \end{vmatrix} = 424 > 0,$$

$$\Delta_2 = \begin{vmatrix} 5 & 1 \\ 19 & 13 \end{vmatrix} = 46 > 0, \qquad \Delta_1 = 5 > 0,$$

i.e. $\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_3 > 0$, $\Delta_4 > 0$. Hence the trivial solution $y \equiv 0$ of equation (5) is asymptotically stable.

Computation can be carried out as follows, for example. We first compose the highest Hurwitz minor Δ_n. Using it, it is easy to write out all the lower minors $\Delta_{n-1}, \ldots, \Delta_1$. We then begin to compute Δ_1, Δ_2, etc. in succession. If we come across a negative minor, the solution is unstable and further computation is unnecessary.

Investigate the zero solution of the following equations for stability:

926. $y''' - 3y' + 2y = 0$.

927. $y^{\mathrm{IV}} + 4y''' + 7y'' + 6y' + 2y = 0$.

928. $y''' + 5y'' + 9y' + 5y = 0.$

929. $y^{IV} - 2y''' + y'' + 2y' - 2y = 0.$

930. $y^{IV} + 7y''' + 17y'' + 17y' + 6y = 0.$

931. $y''' - 3y'' + 12y' - 10y = 0.$

932. $y^{IV} + 5y''' + 18y'' + 34y' + 20y = 0.$

933. $y^{IV} + 7y''' + 19y'' + 23y' + 10y = 0.$

934. $y^{IV} + 11y''' + 41y'' + 61y' + 30y = 0.$

935. $y^{V} + 3y^{IV} - 5y''' - 15y'' + 4y' + 12y = 0.$

936. $y^{V} + 7y^{IV} + 33y''' + 88y'' + 122y' + 60y = 0.$

For what values of α is the zero solution of the following equations stable?

937. $y''' + 2y'' + \alpha y' + 3y = 0.$

938. $y^{IV} + \alpha y''' + 2y'' + y' + 3y = 0.$

939. $y^{IV} + 2y''' + \alpha y'' + y' + y = 0.$

Find for what values of α, β the zero solution of the following equations is stable:

940. $y''' + \alpha y'' + \beta y' + y = 0.$

941. $y^{IV} + 3y''' + \alpha y'' + 2y' + \beta y = 0.$

31. The geometrical criterion of stability (the Mikhailov criterion)

Consider an nth order linear differential equation with constant real coefficients

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0. \qquad (1)$$

Its characteristic equation is

$$f(\lambda) \equiv a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0. \qquad (2)$$

The Mikhailov criterion allows us to solve the question about the location of the roots of the characteristic equation (2) in the complex plane and hence the question about the stability of the zero solution of equation (1). Setting $\lambda = i\omega$ we get

$$f(i\omega) = u(\omega) + iv(\omega),$$

where

$$u(\omega) = a_n - a_{n-2}\omega^2 + a_{n-4}\omega^4 - \ldots,$$
$$v(\omega) = a_{n-1}\omega - a_{n-3}\omega^3 + \ldots.$$

For a given value of the parameter ω the quantity $f(i\omega)$ can be represented as a vector in the u, v complex plane with the beginning at the origin of coordinates.

As ω changes in the interval $(-\infty, +\infty)$ the end of the vector describes some curve, the so-called *Mikhailov curve* (Fig. 45). Since the function $u(\omega)$ is even, the Mikhailov

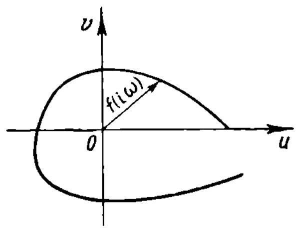

Fig. 45

curve is symmetrical with respect to the Ou axis and so it is sufficient to construct that portion of the curve which corresponds to the change of the parameter ω from 0 to $+\infty$.

If the polynomial $f(\lambda)$ of degree n has m roots with a positive real part and $n - m$ roots with a negative real part, then for the change of ω from 0 to $+\infty$ the angle of rotation φ of the vector $f(i\omega)$ is equal to $\varphi = (n - 2m)\frac{\pi}{2}$.

It is clear that for a solution of equation (1) to be stable it is necessary and sufficient that $m = 0$.

The Mikhailov criterion. *For the zero solution* $y \equiv 0$ *of equation* (1) *to be stable it is necessary and sufficient that* (i) *for the change of* ω *from* 0 *to* $+\infty$ *the vector* $f(i\omega)$ *should rotate through an angle* $\varphi = n\frac{\pi}{2}$, *i.e. it should make* $\frac{\pi}{4}$ *rotations counterclockwise*;

(ii) *the locus of* $f(i\omega)$ *should not pass through the origin* (0, 0) *as* ω *changes from* 0 *to* $+\infty$.

It follows that for a solution of equation (1) to be stable it is *necessary* that all roots of the equations $u(\omega) = 0$, $v(\omega) = 0$ should be real and alternating, i.e. between any

two roots of one equation there must be a root of the other equation.

Example. Investigate the zero solution $y \equiv 0$ of the equation $y^{IV} + y''' + 4y'' + y' + y = 0$ for stability.

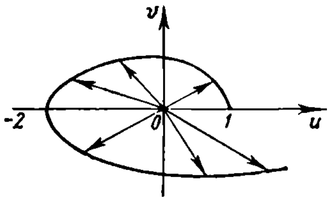

Fig. 46

Solution. We set up the characteristic polynomial

$$f(\lambda) = \lambda^4 + \lambda^3 + 4\lambda^2 + \lambda + 1.$$

Further,

$$f(i\omega) = \omega^4 - i\omega^3 - 4\omega^2 + i\omega + 1, \quad u(\omega) = \omega^4 - 4\omega^2 + 1, \quad v(\omega) = -\omega^3 + \omega = \omega(1-\omega)(1+\omega).$$

We construct the curve (Fig. 46)

$$\begin{cases} u = u(\omega), \\ v = v(\omega), \quad 0 \leqslant \omega < +\infty \end{cases}$$

ω	0	$\sqrt{2-\sqrt{3}}$	1	$\sqrt{2+\sqrt{3}}$
u	1	0	-2	0
v	0	+	0	−

$$\lim_{\omega \to +\infty} \frac{v}{u} = 0.$$

The angle of rotation of the radius vector is $\varphi = 4\frac{\pi}{2} = (n - 2m)\frac{\pi}{2}$. Hence $n - 2m = 4$ and, since $n = 4$, we have $m = 0$, i.e. all roots of the characteristic equation are in the left-hand half-plane. So the trivial solution $y \equiv 0$ is asymptotically stable.

Investigate the zero solution of the following equations for stability using the Mikhailov criterion:

942. $2y''' + 7y'' + 7y' + 2y = 0.$

943. $y''' + 2y'' + 2y' + y = 0.$

944. $2y^{IV} + 13y''' + 28y'' + 23y' + 6y = 0.$

945. $3y^{IV} + 13y''' + 19y'' + 11y' + 2y = 0.$

946. $2y^{IV} + 6y''' + 9y'' + 6y' + 2y = 0.$

947. $y^{IV} + 4y''' + 16y'' + 24y' + 20y = 0.$

948. $y^{V} + 13y^{IV} + 43y''' + 51y'' + 40y' + 12y = 0.$

949. $y''' + y = 0.$

950. $y^{IV} + y''' + y' + y = 0.$

951. $y^{V} + 3y^{IV} + 2y''' + y'' + 3y' + 2y = 0.$

952. $y^{V} + y^{IV} + y''' + y'' + y' + y = 0.$

953. $2y^{IV} + 11y''' + 21y'' + 16y' + 4y = 0.$

954. $y^{VI} + y^{V} + y^{IV} + y'' + y' + y = 0.$

955. $2y^{IV} + 9y''' + 32y'' + 54y' + 20y = 0.$

956. $6y^{IV} + 29y''' + 45y'' + 24y' + 4y = 0.$

957. $y^{V} + y^{IV} + 2y''' + 2y'' + 2y' + 2y = 0.$

958. $y^{VI} + y^{V} + 3y^{IV} + 2y''' + 4y'' + 2y' + 2y = 0.$

959. $y^{V} + 2y^{IV} + y''' + 2y'' + y' + 2y = 0.$

32. Equations with a small parameter of the derivative

Consider a differential equation

$$\frac{dx}{dt} = F(t, x(t), \varepsilon), \tag{1}$$

where ε is a parameter.

If in some closed domain of t, x, ε the function $F(t, x, \varepsilon)$ is continuous with respect to the aggregate of variables

and satisfies a Lipschitz condition in x:

$$| F(t, x_2, \varepsilon) - F(t, x_1, \varepsilon) | \leqslant N | x_2 - x_1 |,$$

N being independent of t, x, ε, then the solution of equation (1) is *continuously* dependent on ε.

In many physical problems one has to consider equations of the form

$$\varepsilon \frac{dx}{dt} = f(t, x), \tag{2}$$

ε being a small parameter.

On dividing both sides of equation (2) by ε we reduce it to the form

$$\frac{dx}{dt} = \frac{1}{\varepsilon} f(t, x), \tag{3}$$

from which it is obvious that the right-hand side of (3) becomes discontinuous when $\varepsilon = 0$, so that the theorem on the continuous dependence of solutions on the parameter ε cannot be made use of in this case.

We ask the question: under what conditions for small values of $|\varepsilon|$ is it possible to discard the term $\varepsilon \frac{dx}{dt}$ from equation (2) and consider the solution of the so-called "degenerate equation"

$$f(t, x) = 0 \tag{4}$$

as an approximation to the solution of the differential equation (2). Let, for definiteness, $\varepsilon > 0$ and let the degenerate solution (4) have only one solution $x = \varphi(t)$. Depending on the behaviour of $f(t, x)$ near the solution $x = \varphi(t)$ of equation (4) the solution $x(t, \varepsilon)$ of the differential equation (2) tends to the solution $x = \varphi(t)$ of the degenerate equation or quickly recedes from it as $\varepsilon \to 0$.

In the former case the solution $x = \varphi(t)$ is said to be *stable*, in the latter case it is said to be *unstable*.

That is, if in passing through the graph of the solution $x = \varphi(t)$ of the degenerate equation (4) the function $f(t, x)$ changes the sign from $+$ to $-$ as x increases with t fixed, then the solution of the degenerate equation, $x = \varphi(t)$, is stable and it can replace approximately the solution $x(t, \varepsilon)$ of equation (2) (Fig. 47).

If, however, the function $f(t, x)$ changes the sign from $-$ to $+$, the solution $x = \varphi(t)$ of the degenerate equation

(4) is unstable and so it is impossible to replace the solution $x(t, \varepsilon)$ of the differential equation (2) by the solution of the degenerate equation (4) (Fig. 48).

Sufficient stability or instability conditions are expressed by the following propositions:

1. *If* $\frac{\partial f(t, x)}{\partial x} < 0$ *for the solution* $x = \varphi(t)$ *of equation* (4), *then the solution* $x = \varphi(t)$ *of the degenerate equation is stable.*

2. *If* $\frac{\partial f(t, x)}{\partial x} > 0$ *for the solution* $x = \varphi(t)$ *of equation* (4), *then the solution* $x = \varphi(x)$ *of the degenerate equation is unstable.*

If the degenerate equation $f(t, x) = 0$ (4) has several solutions $x = \varphi_i(t)$, $i = 1, 2, \ldots, m$, then each of them

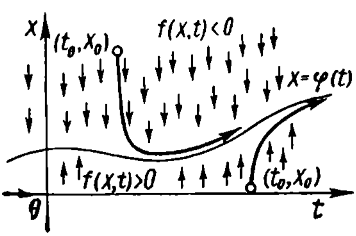

Fig. 47

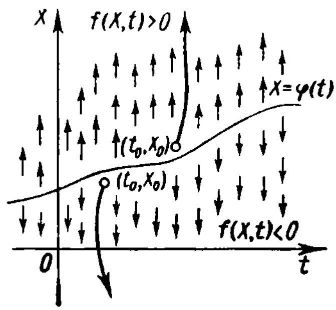

Fig. 48

must be investigated for stability. Here, when $\varepsilon \to 0$, the behaviour of integral curves of equation (2) may differ depending on the choice of initial conditions, the starting point (t_0, x_0).

A semistable case where the function $f(t, x)$ does not change the sign in passing through the curve $x = \varphi(t)$ is also possible (for example, if $x = \varphi(t)$ is a root of even multiplicity of the degenerate equation (4)). In this case, when ε is small, integral curves of equation (2) tend to the curve $x = \varphi(t)$ on one side of the curve and recede from it on the other.

In the former case the starting point (t_0, x_0) is said to belong to the domain of attraction of the semistable solution

$x = \varphi(t)$ and in the latter case it is said to belong to the domain of repulsion.

In a semistable case it is as a rule impossible to replace the solution of the original equation (2) by the solution of the degenerate equation (4).

One can point out criteria for the integral curves of equation (2) to approach the solution $x = \varphi(t)$ of the degenerate equation, with the starting point (t_0, x_0) appropri-

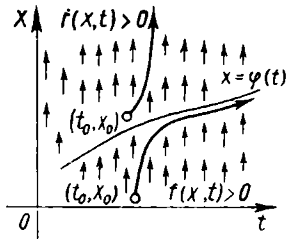

Fig. 49

ately chosen, and to remain in its neighbourhood when $t > t_0$; however this takes place only in the absence of perturbations of equation (2).

These criteria are as follows.

Let the function $f(t, x) \geqslant 0$ in the neighbourhood of the semistable solution $x = \varphi(t)$ of the degenerate equation (4). If $\varphi'(t) > 0$, then the integral curves approaching the curve $x = \varphi(t)$ cannot intersect the curve and remain in its neighbourhood when $t > t_0$ (the starting point (t_0, x_0) must be in the domain of attraction of the semistable solution $x = \varphi(t)$; if (t_0, x_0) is in the domain of repulsion, then the corresponding integral curve of equation (2) quickly recedes from the curve $x = \varphi(t)$) (Fig. 49). If $\varphi'(t) < 0$, then the integral curves approaching the graph of the function $x = \varphi(t)$ will intersect it and quickly recede from the curve $x = \varphi(t)$ on the other side of it. If $\varphi'(t) > 0$ when $t_0 \leqslant t < t_1$ and if $\varphi'(t) < 0$ when $t > t_1$, then for a sufficiently small ε the integral curves emanating from a point (t_0, x_0) belonging to the domain of attraction of the root $x = \varphi(t)$ remain near the curve $x = \varphi(t)$ when $t_0 + \delta < t < t_1$, $\delta > 0$; they intersect the curve $x = \varphi(t)$

in the neighbourhood of the point $t = t_1$ and then recede from it.

If the function $f(t, x) \leqslant 0$ in the neighbourhood of the semistable solution $x = \varphi(t)$, then for the above assertions to be valid the signs of the derivative $\varphi'(t)$ must be reversed.

Example 1. Find out if the solution $x = x(t, \varepsilon)$ of the equation $\varepsilon \frac{dx}{dt} = t^2 - x$ (5), $\varepsilon > 0$, satisfying the initial condition $x|_{t=t_0} = x_0$ tends to the solution of the degenerate equation $x = t^2$ when $t > t_0$ and $\varepsilon \to 0$.

Solution. We have $\frac{\partial f(t, x)}{\partial x} = \frac{\partial (t^2 - x)}{\partial x} = -1 < 0$, so that the solution of the degenerate equation $x = t^2$ is stable

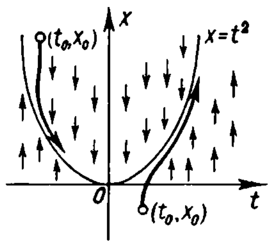

Fig. 50

and the solution of the original equation $x = x(t, \varepsilon)$ emanating from any starting point (t_0, x_0) thus tends to the solution of the degenerate equation when $\varepsilon \to 0$ and $t > t_0$ (Fig. 50).

This can be verified by a direct check. Solving the differential equation (5) as a nonhomogeneous linear equation under the given initial condition $x|_{t=t_0} = x_0$ we find that

$$x(t, \varepsilon) = (x_0 - t_0^2 + 2\varepsilon t_0 - 2\varepsilon^2)\, e^{-\frac{t-t_0}{\varepsilon}} + t^2 - 2\varepsilon t + 2\varepsilon^2,$$

whence it is immediately seen that when $t > t_0$, i.e. $t - t_0 > 0$ and $\varepsilon \to 0$ we have $x(t, \varepsilon) \to t^2$.

Example 2. Investigate the solution of the degenerate equation corresponding to the equation

$$\varepsilon \frac{dx}{dt} = x(e^x - 2)$$

for stability.

Solution. The degenerate equation $x\,(e^x - 2)$ has two solutions

(1) $x = 0$, (2) $x = \ln 2$.

We have

$$\left.\frac{\partial f(t, x)}{\partial x}\right|_{x=0} = (e^x - 2 + xe^x)\,|_{x=0} = -1,$$

so that the solution $x = 0$ is stable;

$$\left.\frac{\partial f(t, x)}{\partial x}\right|_{x=\ln 2} = (e^x - 2 + xe^x)\,|_{x=\ln 2} = 2 \ln 2 > 0,$$

so that the solution $x = \ln 2$ of the degenerate equation is unstable (Fig. 51).

Example 3. Investigate for stability the solution of the degenerate equation corresponding to the equation $\varepsilon \frac{dx}{dt} = (x - t)^2$. The degenerate equation $(x - t)^2 = 0$ has the root $x = t$ of the second multiplicity. The function

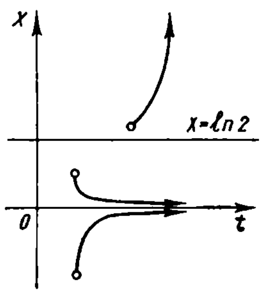

Fig. 51

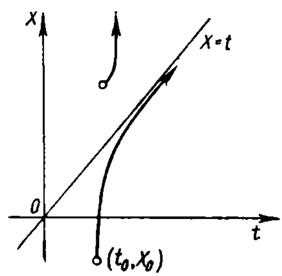

Fig. 52

$f(t, x) \equiv (x - t)^2 > 0$ in the neighbourhood of the root, $\varphi(t) = t$ and $\varphi'(t) = 1 > 0$. Therefore the solution $x = t$ is semistable and if the starting point (t_0, x_0) lies in the half-plane under the straight line $x = t$ (the domain of attraction of the root $x = t$), then the integral curve $x = x(t, \varepsilon)$ emanating from the point (t_0, x_0) will remain in the neighbourhood of the line $x = t$ when $t > t_0$ (Fig. 52).

Investigate for stability the solutions of the degenerate equations corresponding to the following differential equa-

tions:

960. $\varepsilon \frac{dx}{dt} = x - t^2$.

961. $\varepsilon \frac{dx}{dt} = x(t^4 + 1 - x)$.

962. $\varepsilon \frac{dx}{dt} = (x - t)(x - e^t)$.

963. $\varepsilon \frac{dx}{dt} = x^2 - t^2$.

964. $\varepsilon \frac{dx}{dt} = xt$.

965. $\varepsilon \frac{dx}{dt} = (x - t)(\ln x - t^2 - 1)$.

966. $\varepsilon \frac{dx}{dt} = (t + x)^2$.

967. $\varepsilon \frac{dx}{dt} = x - t + 1$.

ANSWERS

1. If $y(x)$ is a solution of a differential equation, then it turns the equation into an identity. Therefore, if equations (a) and (b) have coinciding solutions, then their left- and right-hand sides are identically equal: $y^2 + 2x - x^4 = -y^2 - y + 2x + x^2 + x^4$. Hence we find that $y = x^2$, $y = -x^2 - \frac{1}{2}$. The second function does not satisfy equation (a) and so must be discarded. We get $y = x^2$. 13. $y \equiv 0$. 17. $\alpha = \arctan \frac{1}{2}$. 18. $\alpha = \frac{\pi}{4}$. 19. The extreme points of the integral curves are on the straight line $x = -1$. 20. The inflection points of the integral curves are on the parabola $y = x^2 + 2x$.

41. $y_0(x) = 0,\ y_1(x) = \dfrac{1+x^3}{3}$,

$$y_2(x) = \frac{1}{126}(33 - 14x + 42x^3 - 7x^4 - 2x^7).$$

42. $y_0(x) = 0,\ y_1(x) = \dfrac{x^2}{2},\ y_2(x) = \dfrac{x^2}{2} + \dfrac{x^5}{20}$.

43. $y_0(x) = 1,\ y_1(x) = 1 + x + \dfrac{x^2}{2},\ y_2(x) = 1 + x + x^2 + \dfrac{x^3}{6}$.

44. $y_0(x) = 2,\ y_1(x) = 2 + x - \dfrac{2}{3}x^3$,

$$y_2(x) = 2 + x + x^2 - \frac{2}{3}x^3 - \frac{1}{4}x^4.$$

45. $y_0(x) = 2,\quad y_1(x) = 2x - \ln x,\quad y_2(x) = 2 + \ln^2 x$.
46. $\arctan x + \arctan y = \tilde{C}$ or $x + y = C(1 - xy)$.
47. $x^2(1 + y^2) = C$. 48. $y = \sin x$. 49. $y = \tan \ln Cx$.

21.

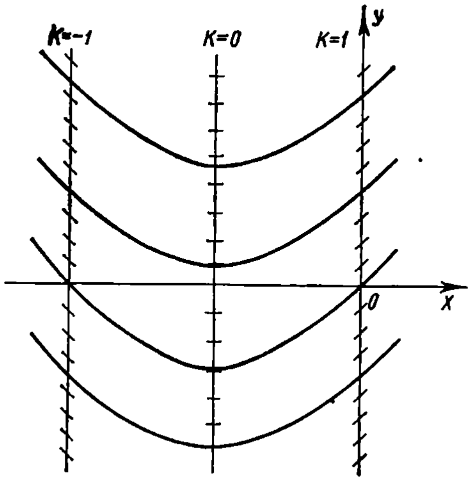

Fig. 53

22.

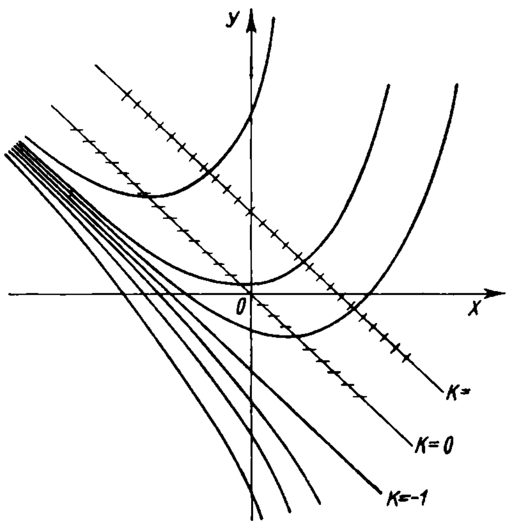

Fig. 54

23.

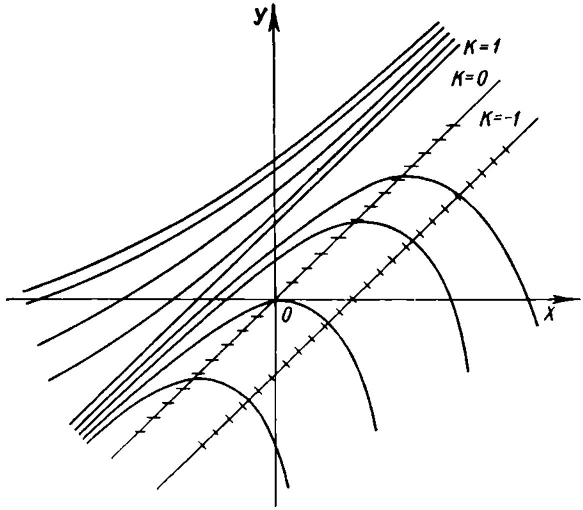

Fig. 55

24.

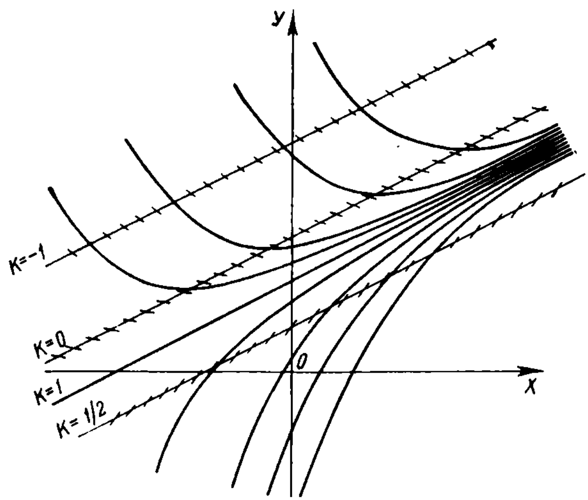

Fig. 56

25.

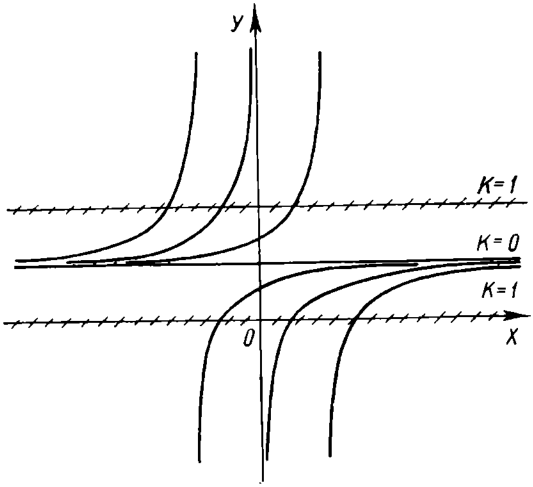

Fig. 57

26.

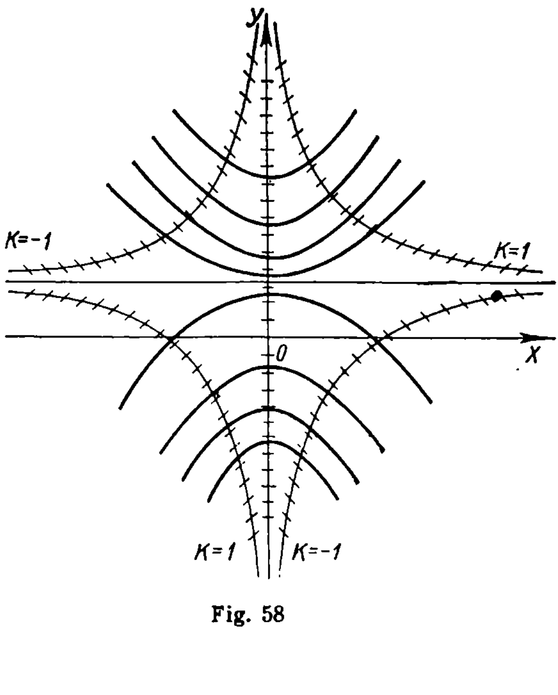

Fig. 58

27.

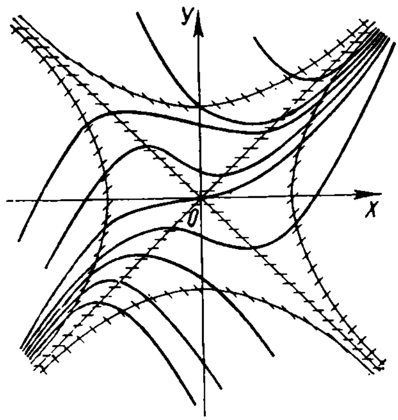

Fig. 59

28.

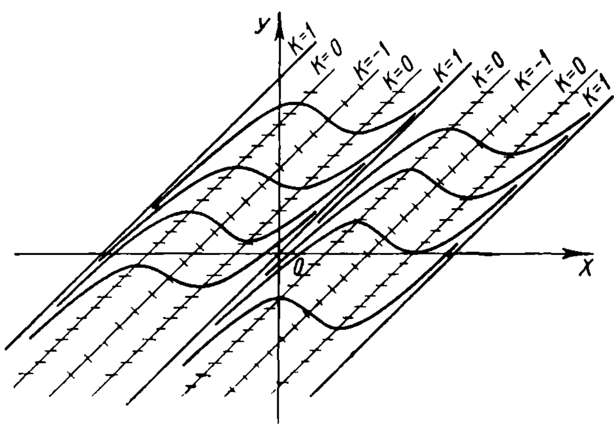

Fig. 60

29.

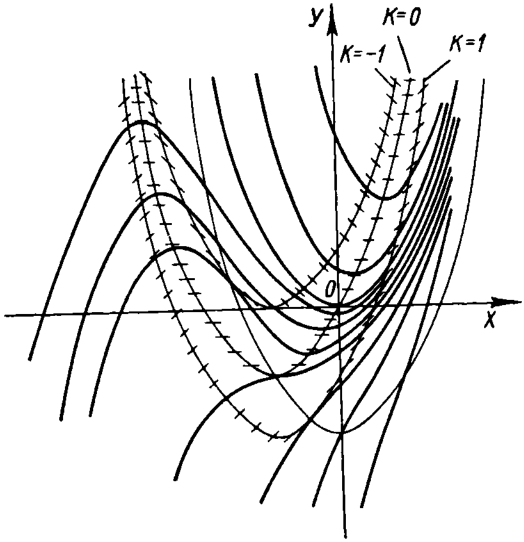

Fig. 61

30.

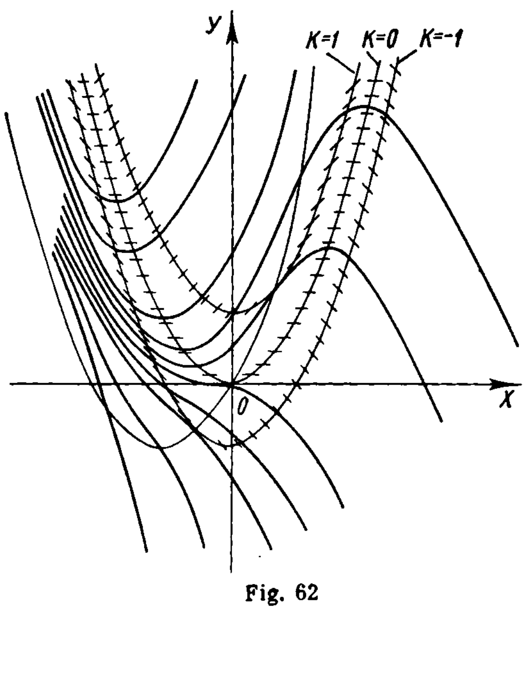

Fig. 62

31.

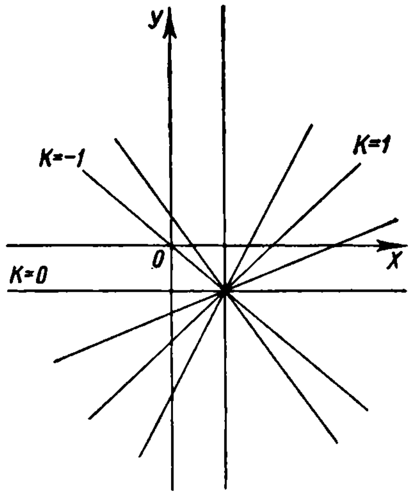

Fig. 63

32.

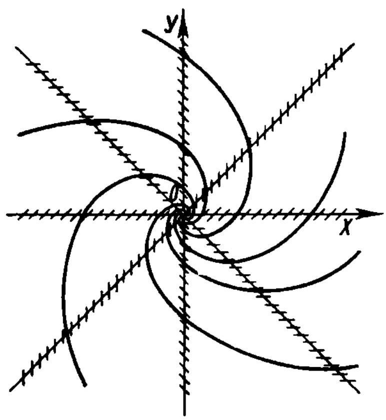

Fig. 64

50. $\sqrt{1+x^2}+\sqrt{1+y^2}=C$. 51. $\sqrt{1-x^2}+\sqrt{1-y^2}=1$, $y=1$. 52. $e^x=C(1-e^{-y})$.

53. $y=1$. 54. $a^x+a^{-y}=C$. 55. $1+e^y=C(1+x^2)$.

56. $y=\sin[C+\ln(1+x^2)]$. 57. $\arctan e^x=\frac{1}{2\sin^2 y}+C$.

58. $y=(1+Cy+\ln y)\cos x$. 59. $x+C=\cot\left(\frac{y-x}{2}+\frac{\pi}{4}\right)$.

60. $b(ax+by+c)+a=Ce^{bx}$.

61. $x+y=a\tan\left(C+\frac{y}{a}\right)$. 62. $y=-\frac{1}{x}$.

63. $y=a\tan\sqrt{\frac{a}{x}-1}$. 64. $\tan\frac{y}{2}=e^{2\sin x}$.

65. $y'=3y$, $y=-2e^{3x}$.

66. $\int_0^x y\,dt=a^2\ln\frac{y}{a}$; $y=\frac{a^2}{a-x}$ (the hyperbola).

67. $\frac{dv}{dt}=20\frac{t}{v}$, $v=50\sqrt{29}$ cm/s.

69. $m\frac{dv}{dt}=kv^2$, $t=\frac{h(v_1-v_0)}{v_0v_1\ln\frac{v_1}{v_0}}=\frac{3}{40\ln 2.5}$ s.

70. $m\frac{dv}{dt}=-kv$; $t=-\frac{5\ln 10}{\ln 0.8}$ s. 72. $\frac{dT}{dt}=k(T-T_0)$; $T=20+80\left(\frac{1}{2}\right)^{t/20}$; $t=60$ min.

73. $y'=n\frac{y}{x}$; $y=Cx^n$. 74. $\frac{dS}{dt}=kS$; $S=25\times 2^{t/5}$.

75. 18.1 kg; $\frac{dx}{dt}=k\left(\frac{1}{3}-\frac{x}{300}\right)$, k being the proportionality coefficient. 76. 5.2 kg; $\frac{dx}{dt}=kx\left(\frac{10-x}{90}-\frac{1}{3}\right)$.

77. $xy=C$ $(C\neq 0)$. 78. 0.82 kg; $\frac{ds}{dt}=ks(s+6)$.

79. 32.2 min. 80. $T=\frac{2}{3}x$; $864{,}000\times 4.2$ J; $\frac{dT}{dx}=-\frac{Q}{kS}$, where $Q=\text{const}$.

83. $y=0$, for $\alpha\leqslant 1$ the solution is unique.

85. $y=\frac{\pi}{2}(2n+1)x+C$, $n=0, \pm 1, \pm 2, \ldots$.

86. $y=C$. 87. $y=(-1)^n\left(x\arcsin x+\sqrt{1-x^2}\right)+n\pi x+C$, $n=0, \pm 1, \pm 2, \ldots$ 88. $y=e^x+C$.

89. $y=n\pi x+C$, $n=0, \pm 1, \pm 2, \ldots$.

90. $y=x(\ln x-1)+C$. 91. $y=x\arctan x-\frac{1}{2}\ln(1+x^2)+n\pi x+C$, $n=0, \pm 1, \pm 2, \ldots$.

92. $y=\arcsin\left(\frac{\sqrt{3}}{2}-\frac{1}{x}\right)+5\pi$.

93. $y=\operatorname{arccot}\left(\frac{2}{x}+\frac{1}{\sqrt{3}}\right)+3\pi$.

94. $y=2\arctan\left(1-\frac{1}{2x^2}\right)+\frac{9\pi}{2}$.

95. $y=\frac{1}{2}\arctan\left(\frac{\pi}{2}+\arctan x\right)+\frac{7\pi}{2}$.

96. $y=0$. 97. $y=1$. 98. $y=-\pi$.

99. $y=\operatorname{arccot}\frac{1}{2x}+\frac{9}{4}\pi$. 100. $\tan\frac{y}{x}=\ln Cx$.

101. $y=x(C-\ln x)$. 102. $y=xe^{1+Cx}$.

103. $(x-y)\ln Cx=x$. 104. $y+\sqrt{y^2-x^2}=Cx^2$, $y=x$, $y=-x$. 105. $2x=(x-y)\ln Cx$.

106. $y^2-3xy+2x^2=C$. 107. $y^2+2xy-x^2=C$.

108. $y=1+(x-1)\ln C(x-1)$.

109. $(x+y-1)^5(x-y-1)^3=C$.

110. $y^2-2xy-x^2-8y+4x=C$.

111. $y^2-2xy-x^2+4y=C$.

112. $y^2+3xy+x^2-5x-5y=C$.

113. $(4x+2y+1)^2=4x+C$.

114. $x+3y-\ln|x-2y|=C$.

115. $(x+y-1)^2+2x=C$. 116. $y^2=x\ln Cy^2$.

117. $Cx^4=y^6+x^3$. 118. $\sqrt{x^2y^4+1}=Cx^2y^2-1$.

119. $2\arctan\frac{y^3}{x}=\ln(x^2+y^6)+C$. 120. $x^2+y^2=Cx$.

121. $y=\frac{1}{2}\left(Cx^{1-k}-\frac{1}{C}x^{1+k}\right)$.

122. $y^2=2Cx+C^2$. 123. $y=\frac{1}{2}\left(Cx^2-\frac{1}{C}\right)$.

124. $x^2+y^2=Cx^4$. 125. $y=Ce^{-2x}+e^{-x}$.

126. $y=x-x^2$. 127. $y=(C+x^2)e^{x^2}$.

128. $y=(C+x)e^{-x^2}$. 129. $y=\frac{x^2}{\cos x}$.

130. $y=Cx^2+x^2\sin x$. 131. $y=\frac{\sin x}{\cos^2 x}$.

132. $y=(C+x^3)\ln x$. 133. $x=Cy-\frac{y^2}{2}$. 134. $y=1$.

135. $x=\frac{C}{y}+y\ln y$. 136. $x=(C+y)e^{-\frac{y^2}{2}}$.

137. $y=(C+x^2)e^{e^x}$. 138. $y=(C+x)e^{(1-x)e^x}$.

139. $i(t)=\frac{E_0}{R^2+(2n\pi L)^2}\left[R\sin 2n\pi t+2n\pi L\left(e^{-\frac{Rt}{L}}-\cos 2n\pi t\right)\right]$ $+I_0e^{-\frac{Rt}{L}}$. 140. $q=QE\left(1-e^{-t/QR}\right)$; $R\frac{dq}{dt}=E-\frac{q}{Q}$.

141. $v=\frac{k_1}{k_2}\left(t-\frac{m}{k_2}+\frac{m}{k_2}e^{-\frac{k_2t}{m}}\right)$; $m\frac{dv}{dt}=k_1t-k_2v$, $v(0)=0$.

142. $y=Cx-x^2$; $y-xy'=x^2$. 143. $y=C\sqrt{x}-x$; $x-xy'=\frac{x+y}{2}$. 144. $y(x)=y_1(x)+Ce^{-\int p(x)dx}$.

145. $y=y_1(x)+C[y_2(x)-y_1(x)]$. 148. $y=2^{\sin x}$.

149. $y=e^{-x}$. 150. $y=\frac{\sin x}{x}$. 151. $y=\frac{x+1}{x\cos\frac{1}{x}}$.

152. $y=\frac{1}{\sqrt{x}}-1$. 153. $y=e^x+e^{\frac{1}{x}}$. 154. $y=x$.

155. $y=\frac{x}{\sin x}$. 156. $y=\cos x$. 157. $y=\frac{1}{1+Ce^{x^2}}$.

158. $y^3=x^3+Cx^3$. 159. $x^3e^{-y}=C+y$. 160. $y=\frac{e^{-x^2}}{C-x}$.

161. $\sqrt{y}+1=Ce^{e^x}$. 162. $y^2\ln x=C+\sin x$.

163. $y^2(C-x)\sin x=1$. 164. $y^4+2x^2y^2+2y^2=C$.

165. $y=\dfrac{1}{Ce^{-\sin x}-1}$. **166.** $\sin y=(x+C)\,e^{x}$, $z=\sin y$.

167. $\ln y=(x+C)\,e^{x}$, $z=\ln y$. **168.** $\sin y=x+C\,e^{-x}$, $z=\sin y$. **169.** $x-2+C\,e^{-x}=e^{y^2/2}$, $z=e^{y^2/2}$.

170. $\tan y=(C+x^2)\,e^{-x^2}$; $z=\tan y$. **171.** $xy=C$.

172. $y=(x+1)\,e^{x}$. **173.** $y=2-(2+a^2)\,e^{\frac{x^2-a^2}{2}}$.

174. $y=Cx^{\frac{1-n}{n}}$. **175.** $x^4+x^2y^2+y^4=C$.

176. $x^3+3x^2y^2+y^4=C$. **177.** $\sqrt{x^2+y^2}+\ln|xy|+\dfrac{x}{y}=C$. **178.** $x^3\tan y+y^4+\dfrac{y^3}{x^2}=C$.

179. $x^3y+x^2-y^2=Cxy$. **180.** $\dfrac{\sin^2 x}{y}+\dfrac{x^2+y^2}{2}=C$.

181. $x^3+y^3-x^2-xy+y^2=C$. **182.** $y\sqrt{1+x^2}+x^2y-y\ln x=C$. **183.** $\sqrt{x^2+y^2}+\dfrac{y}{x}=C$.

184. $x\sin y-y\cos x+\ln|xy|=C$. **185.** $\tan xy-\cos x-\cos y=C$.

186. $y=x$. **187.** $(x^2+y^2)^2+2a^2(y^2-x^2)=C$.

188. $xy(x^2+y^2)=C$. **189.** $xy^2-2x^2y-2=Cx$; $\mu=1/x^2$.

190. $x-\dfrac{y}{x}=C$; $\mu=\dfrac{1}{x^2}$. **191.** $x\ln|x|-y^2=Cx$; $\mu=1/x^2$.

192. $5\arctan x+2xy=C$; $x=0$; $\mu=\dfrac{1}{1+x^2}$.

193. $y^3+x^3(\ln x-1)=Cx^2$; $\mu=1/x^4$.

194. $2e^{x}\sin y+2\,e^{x}(x-1)+e^{x}(\sin x-\cos x)=C$; $\mu=e^{x}$.

195. $x^2-\dfrac{7}{y}-3xy=C$; $\mu=1/y^2$. **196.** $(x+y^2)^2\,C=x-y^2$; $\mu=\dfrac{1}{(x+y^2)^2}$. **197.** $1+y^2-x^2=Cx$; $\mu_2=1/x^2$; $\mu_1=\dfrac{1}{(1+y^2-x^2)^2}$. **198.** $y-1=C\sqrt{x^2+y^2}$; $\mu=(x^2+y^2)^{-\frac{3}{2}}$. **199.** $(y-C)^2=x^3$.

200. $\ln Cy=x\pm 2e^{\frac{x}{2}}$, $y=0$. **201.** $y=2x^2+C$, $y=-x^2+C$.

202. $xy=C$, $x^2y=C$. **203.** $y=\dfrac{x^2}{2}+C$, $y=C\,e^{x}-x-1$.

204. $4e^{-y/3}=(x+2)^{4/3}+C$. **205.** $y=\frac{x^2}{2}+C$.

$y=-\frac{x^2}{2}+C$, $y=C\,e^x$. **206.** $y=C\,e^x+\frac{1}{C}$,

$y=\pm 2e^{x/2}$.

207. $y=Cx+\frac{1}{2}(x^2-C^2)$, $y=x^2$.

208. $\begin{cases} x=e^p(p+1)+C, \\ y=p^2e^p, \ y=0. \end{cases}$ **209.** $\begin{cases} x=\ln|\ln p|+\frac{1}{\ln p}+C, \\ y=\frac{p}{\ln p}. \end{cases}$

210. $\begin{cases} x=\ln p+\sin p, \\ y=C+p(1+\sin p)+\cos p. \end{cases}$

211. $\begin{cases} x=p^2-2p+2, \\ y+C=\frac{2}{3}p^3-p^2. \end{cases}$ **212.** $\begin{cases} x+c=\frac{(\ln p+1)^2}{2}, \\ y=p\ln p. \end{cases}$

213. $\begin{cases} x=e^p+C, \\ y=(p-1)e^p; \ y=-1. \end{cases}$ **214.** $\begin{cases} x=\frac{e^{1/p}}{p^2}, \\ y=C+e^{1/p}\left(1+\frac{1}{p}\right). \end{cases}$

215. $\begin{cases} x=a\cos^3 t, \\ y+C=-a\sin^3 t, \ p=\tan t. \end{cases}$

216. $\begin{cases} x=5\left\{\frac{1}{3}\tan^3 t-\tan t+t\right)+C, \\ y=a\sin^5 t. \end{cases}$

217. $\begin{cases} x=p+\sin p, \\ y+C=\frac{1}{2}p^2+p\sin p+\cos p. \end{cases}$

218. $\begin{cases} x+C=\ln|p|+\sin p+p\cos p, \\ y=p+p^2\cos p. \end{cases}$

219. $\begin{cases} x+C=2\arctan p-\ln\left|\frac{1+\sqrt{1-p^2}}{p}\right|, \\ y=\arcsin p+\ln(1+p^2), \end{cases}$
$y=0$.

220. $\begin{cases} x=\frac{C}{p^2}-\frac{1}{p}, \\ y=\frac{2C}{p}+\ln p-2. \end{cases}$

221. $\begin{cases} x = 2(1-p) + Ce^{-p}, \\ y = [2(1-p) + Ce^{-p}](1+p) + p^2. \end{cases}$

222. $\begin{cases} x = \dfrac{C}{p^2} - \dfrac{\cos p}{p^2} - \dfrac{\sin p}{p}, \\ y = \dfrac{2C}{p} - \dfrac{2\cos p}{p} - \sin p. \end{cases}$

$y = 0.$

223. $\begin{cases} x = \dfrac{Cp^2 + 2p - 1}{2p^2(p-1)^2}, \\ y = \dfrac{Cp^2 + 2p - 1}{2(p-1)^2} - \dfrac{1}{p}. \end{cases}$

224. $\begin{cases} x = \dfrac{C}{p^2} - 2e^p\left(\dfrac{1}{p} - \dfrac{2}{p^2} + \dfrac{2}{p^3}\right), \\ y = \dfrac{3C}{2p^2} - 2e^p\left(1 - \dfrac{3}{p} + \dfrac{3}{p^2}\right). \end{cases}$

225. $y = Cx + \frac{a}{C^2}$; $4y^3 = 27ax^2$. 226. $y = Cx + C^2$, $y = -\frac{x^2}{4}$. 227. $y = Cx - \frac{C-1}{C}$; $(y+1)^2 = 4x$.

228. $y = Cx + a\sqrt{1 + C^2}$; $x^2 + y^2 = a^2$.

229. $x = Cy + C^2$; $4x = -y^2$. 230. $xy = \pm a^2$.

231. $x^{2/3} + y^{2/3} = a^{2/3}$. 232. $y = e^x + \frac{1}{C + e^x}$.

233. $y = \sin x + \frac{1}{C + x}$. 234. $y = x + \frac{1}{Cx + 1}$.

235. $y = -\frac{1}{x} + \frac{1}{(C - \ln x)x}$. 236. In this case

$$\frac{dy}{dx} = -[a(x)y^2 + b(x)y + c(x)];$$

$$\frac{dy}{dx} = -c(x)\left[\frac{a(x)}{c(x)}y^2 + \frac{b(x)}{c(x)}y + 1\right];$$

$\dfrac{dy}{dx} = -c(x)\left(\dfrac{m}{p}y^2 + \dfrac{n}{p}y + 1\right)$ and so the variables are separable.

We have: $C - \int c(x)\,dx = \frac{1}{p}\int \frac{dy}{my^2 + ny + p}$.

238. $y + xy' = 0$. 239. $x^2 + y^2 = 2xyy'$.

240. $xy' = y\ln y'$. 241. $y'^2 + y' - xy' + y = 0$.

242. $y'' - 2y' + y = 0$. 243. $yy'^2 + 2xy' = y$.

244. $y''' = 0$. 245. $y''' + \frac{3}{x} y'' = 0$. 246. $y''^2 = (1 + y'^2)^3$.
247. $y'' - y = 0$. 248. $y'' + y = 0$. 249. $2x^2 + y^2 + C$.
250. $x^2 + ny^2 = C$. 251. $2x + \sigma y^2 = C$. 252. $\sin y = Ce^{-x}$.
253. $y^2 = Cx$. 254. $xy = C$. 255. $y = Cx$ if $k = 2$;
$\frac{1}{x^{k-2}} - \frac{1}{y^{k-2}} = \frac{1}{C^{k-2}}$ if $k \neq 2$.
256. $x^2 + y^2 = Cx$. 257. $xy^3 = C$. 258. $\rho = C(1 - \cos \varphi)$.
259. $y = Ce^{-x/2}$. 260. $y^2 = 4x + 4$. 261. $y = 0$.
262. $y = 0$, $y = \frac{4}{27} x^3$. 263. No singular solutions.
264. $a = 0$, $y = 0$. 265. $4y + x^5 = 0$. 266. $4xy^2 + 1 = 0$.
267. $y = x - \frac{4}{27}$. 268. No singular solutions.
269. $y = x^2/4$. 270. $y = 0$; $y = 4x$. 271. $y = \pm 1$.
272. $y = \pm 2e^{x/2}$. 273. $y = x$, $y = -x/3$.
274. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. 275. $y = x - \frac{1}{x + C}$.
276. $y = C \frac{\sin x}{x} + \cos x$. 277. $y^{1-n} = 2 \sin x + \frac{2}{n-1} + Ce^{(n-1)\sin x}$. 278. $x^4 - 6x^2y^2 + y^4 = C$.
279. $15x^2y - 24xy^2 - 12x^3 + 2y^3 = C$.
280. $6y + 12y^3 - 9x^2y^2 + 2x^3 = C$. 281. $2 + xy \ln^2 x = Cxy$; $\mu = \frac{1}{x^2y^2}$. 282. $y = Ce^{-x^2} + (\sin x - x \cos x) e^{-x^2}$.
283. $x = Ce^{2y} + \frac{y^2 + y}{2} + \frac{1}{4}$. 284. $y + C = 2x - \frac{x^2}{2} + 2 \ln |1 - x|$. 285. $y^2 = Cx^2 + x^4$.
286. $y(y - 2x)^3 = C(y - x)^2$. 287. $y = C(2x - 1) + \frac{1}{x}$.
288. $x + y = 1 = Ce^{\frac{2x+2}{x+y-1}}$. 289. $\ln \left| \tan \frac{y}{4} \right| = C - 2 \cos \frac{x}{2}$.
290. $y = C(3x^2 - 2x)$. 291. $y^3 = Cx^3 + x^4$.
292. $x + ye^{\frac{x}{y}} = 1 + e$. 293. $\ln |x| - \frac{y^2}{2x^2} = C$.
294. $\ln |2x - 2y + 5| - 2(x + y - 2) = C$.
295. $x^2y + 2x = Cy$.

296. $x^2+y^2=C\,e^{-x}$. 297. $\frac{1}{2}\ln\frac{x^2+x+1}{y^2-y+1}$

$-\sqrt{3}\left(\arctan\frac{2x+1}{\sqrt{3}}+\arctan\frac{2y-1}{\sqrt{3}}\right)=C.$

298. $x=y^2(1+Ce^{1/y})$. 299. $\sin x+2y\ln|y|-Cy=0$.

300. $3e^{-2y}=C\,e^{-2x}-2e^x$.

301. $x^4+y^2=C(x^2+y)$. 302. $\frac{1}{x}=\frac{C}{y^2}+\frac{y^n}{n+2}$; $n\neq-2$.

303. $7(3y-4x)+(4a^2-3b^2)\ln|7(x+y)+a^2+b^2|=C$.

304. $x\,e^{\frac{y^2}{x}}=C$. 305. $y(1+x+\ln x)=1$.

306. $y[\sin(\ln y)+\cos(\ln y)]=x[\sin(\ln x)-\cos(\ln x)]+C$.

307. $(x-1+\sqrt{x^2-2x+5})^3=C\left(3y-1+\sqrt{9y^2-6y+2}\right)$.

308. $(x-y)(x+7y-4)=C$. 309. $x+2y$
$+3\ln|x+y-2|=5$.

310. $y^2=C\,e^{y^2/x}$. 311. $2\arctan\frac{y+2}{x-3}+\ln C(y+2)=0$.

312. When going over from a curve to a curve symmetrical to it with respect to O (0, 0) the variable x, y, y' are replaced by $-x$, $-y$, y' and the given equation is again satisfied. 313. For the straight line $y = kx + b$ to be an integral curve of the given equation it is necessary and sufficient by virtue of the equation $y' = k$ that $y = kx + b \equiv k + xk^2$, i.e. $k = 0$ or $k = 1$ and $b = k$. Hence we obtain two solutions: $y = 0$ and $y = x + 1$.

314. $3y^2 - 2x = C$. 315. $y = \cosh x$ and $y = 1$. 317. (a) No, they cannot; (b) No, they cannot; (c) Yes, they can.

327. $y=\frac{x^5}{120}+C_1x^3+C_2x^2+C_3x+C_4$.

328. $y=\frac{x^4}{24}-\sin x+C_1x^2+C_2x+C_3$.

329. $y=\frac{1}{12(x+2)^3}$. 330. $y=(x-2)e^x+x+2$.

331. $y=\frac{x^3}{3}\ln x-\frac{5}{18}x^3+C_1x+C_2$.

332. $y=C_1x^2+C_2$. 333. $y=C_1\ln|x|+C_2$.

334. $y=C_1e^{x^2}+C_2$. 335. $y=\frac{x^3}{3}+C_1x^2+C_2$.

336. $y=C_1x(\ln x-1)+C_2$.

337. $y=(C_1x-C_1^2)\,e^{\frac{x}{C_1}+1}+C_2$.

338. $y=\frac{\sqrt{2}}{5}x^{5/2}$. 339. $y=C_3+C_2x-\sin(x+C_1)$.

340. $y=C_1x^3+C_2x+C_3$. 341. $y=\cosh(x+C_1)+C_2$.

342. $y=C_2-\ln|C_1-x|$. 343. $y=C_2-\cos(C_1+x)$.

344. $y=C_2-\ln|\cos(C_1+x)|$. 345. $y=\frac{(x+C_1)^3}{12}-x+C_2$.

346. $y=x$. 347. $y=-2x$. 348. $y=C_2-\ln|1-e^{x+C_1}|$.

349. $(x+C_1)^2+(y+C_2)^2=9$.

350. $y=(x+C_1)\ln|x+C_1|+C_2x+C_3$. 351. $y=C_2e^{C_1x}$.

352. $y=\frac{1}{\sqrt{1-x}}$. 353. $y=\left(1+\frac{x}{3}\right)^3$.

354. $y=\frac{4}{(x+4)^2}$. 355. $y^2=C_1x+C_2$.

356. $y=\frac{1}{C_1}(1+C_2e^{C_1x})$. 357. $y=C_1\cosh\frac{x+C_2}{C_1}$.

358. $y=\frac{1}{C_1}\left[1+\frac{(C_1x+C_2)^2}{4}\right]$. 359. $y=\sqrt{2x-x^2}$.

360. $C_1x+C_2=\ln\left|\frac{y}{y+C_1}\right|$. 361. $y=-\ln|x-1|$.

362. $y\cos^2(x+C_1)=C_2$. 363. $y=\frac{4}{(x-2)^2}$.

364. $(x-C_1)^2-C_2y^2+kC_2^2=0$.

365. It is a parabola. 366. $\frac{d^2x}{dt^2}=-\frac{k}{x^2}$, where x is the distance of the body from the centre of the Earth; $t\approx 122$ hrs.

367. $m\frac{d^2x}{dt^2}=\frac{k}{x^3}$; $x^2=\frac{a^2}{C_1}(t+C_2)^2+C_1$; $a^2=\frac{k}{m}$.

368. $m\frac{d^2x}{dt^2}=mg-k\left(\frac{dx}{dt}\right)^2$;

$$x=\frac{m}{k}\ln\frac{e^{\alpha t}+e^{-\alpha t}}{2}=\frac{m}{k}\ln\cosh\alpha t;\ \alpha=\sqrt{\frac{kg}{m}}.$$

369. $Cx=y^{2k-1}$ $(k>1/2)$. 370. $(x+C_1)^2+(y+C_2)^2=R$, where $R=\text{const}$. 371. Yes. 372. No. 373. No. 374. Yes. 375. Yes. 376. Yes. 377. No. 378. No. 379. No. 380. No. 381. No. 382. No. 383. No. 384. No. 385. Yes. 386. No.

389. 1. **390.** $-\frac{2}{x}(x\neq 0)$. **391.** 0.

392. e^{-2x}. **393.** 0. **394.** $-8\sin^3 x$. **395.** $-1/\sqrt{2}$.

396. $\frac{\pi}{2\sqrt{\pi^2-x^2}}$; $|x|<\pi$. **397.** 0. **398.** 0. **399.** $1-\ln x$; $x>0$. **400.** $\frac{x-1}{x^3}e^{1/x}$; $x\neq 0$. **401.** $-e^{2x}$.

402. $-2e^{-6x}$. **403.** 1. **404.** 1. **410.** $9y''-6y'+y=0$.

411. $y''+3y'+2y=0$. **412.** $2y''-3y'-5y=0$.

413. $y'''+3y''+2y'=0$. **414.** $y^{\mathrm{IV}}+2y''+y=0$. **415.** $y'''=0$.

416. $y''-3y'+2y=0$, $y=C_1e^x+C_2e^{2x}$.

417. $y''-2y'+y=0$; $y=(C_1+C_2x)e^x$.

418. $y''-6y'+13y=0$; $y=e^{3x}(C_1\cos 2x+C_2\sin 2x)$.

419. $y'''-3y''+3y'-y=0$, $y=e^x(C_1+C_2x+C_3x^2)$.

420. $y''-y=0$. **421.** $y''-y'=0$. **422.** $y''+4y'+4y=0$.

423. $y''+9y=0$. **424.** $y''=0$. **425.** $y'''-6y''+11y'-6y=0$.

426. $y'''-3y''+3y'-y=0$. **427.** $y'''-4y''+5y'-2y=0$.

428. $y'''-y''=0$. **429.** $y'''+y'=0$.

430. $y'''-2y''+y'-2y=0$. **431.** $y'''+2y''+2y'=0$.

432. $y=C_1e^x+C_2e^{-x}$. **433.** $y=C_1e^{2x}+C_2e^{-\frac{4}{3}x}$.

434. $y=e^x(1+x)$. **435.** $y=e^{-x}(C_1+C_2x)$.

436. $y=4e^x+2e^{3x}$. **437.** $y=C_1e^{-x}+C_2e^{-2x}+C_3e^{-3x}$.

438. $y=C_1e^{(1-\sqrt{3})x}+C_2e^{(1+\sqrt{3})x}$.

439. $y=C_1+C_2x+C_3x^2+C_4x^3+e^{-x}(C_5+C_6x)$.

440. $y=e^x\left(C_1\cos\frac{x}{2}+C_2\sin\frac{x}{2}\right)$. **441.** $y=C_1e^{2x}$ $+e^{-x}(C_2\cos\sqrt{3x}+C_3\sin\sqrt{3}\,x)$.

442. $y=e^{-x}(C_1+C_2x)+e^{-x}(C_3\cos 2x+C_4\sin 2x)$.

443. $y=e^x\sin x$. **444.** $y=e^x(\cos\sqrt{2}\,x+\sqrt{2}\sin\sqrt{2}\,x)$.

445. $y=C_1e^x+C_2e^{-x}+e^{-x}(C_3\cos 2x+C_4\sin 2x)$.

446. $y=C_1e^x+C_2e^{-2x}+e^{-x}(C_3+C_4\cos x+C_5\sin x)$.

447. $y=C_1e^x+C_2e^{-x}+C_3e^{-2x}$.

448. $y=C_1+e^x(C_2\cos x+C_3\sin x)$. **449.** $y=C_1e^x$ $+C_2e^{-x}+C_3\cos x+C_4\sin x$. **450.** $y=C_1+C_2x$

$+ C_3x^2 + \ldots + C_{10}x^9$. **451.** $y = e^{-x}(C_1 + C_2x) + C_3e^{2x}$.

452. $y = C_1 + C_2e^x + C_3e^{x/2}$. **453.** $y = x + e^{-x}$.

454. $y_{p.n} = A_1x^2 + A_2x + A_3$. **455.** $y_{p\,n} = A_1x^3 + A_2x^2 + A_3x$.

456. $y_{p.n} = A_1x^4 + A_2x^3 + A_3x^2$.

457. $y_{p.n} = e^{-x}(A_1 + A_2x)$.

458. $y_{p.n} = e^{-x}(A_1x + A_2x^2)$.

459. $y_{p.n} = e^{-x}(A_1x^2 + A_2x^3)$.

460. $y_{p.n} = A \sin x + B \cos x$.

461. $y_{p.n} = x(A \sin x + B \cos x)$.

462. $y_{p.n} = x(C_1 \cos 2x + C_2 \sin 2x)$.

463. $y_{p.n} = x(C_1 \cos kx + C_2 \sin kx)$. **464.** $y_{p.n} = e^{-x}(C_1 \cos x + C_2 \sin x)$. **465.** $y_{p.n} = xe^{-x} \times (C_1 \cos x + C_2 \sin x)$.

466. $y_{p.n} = C_1 + C_2x + C_3x^2$. **467.** $y_{p.n} = C_1x + C_2x^2 + C_3x^3$.

468. $y_{p.n} = C_1x^2 + C_2x^3 + C_3x^4$. **469.** $y_{p.n} = C_1x^3 + C_2x^4 + C_3x^5$. **470.** $y_{p.n} = x(C_1 \cos x + C_2 \sin x)$.

471. (a) $y_{p.n} = (A_1 + A_2x + A_3x^2)e^{kx}$, (b) $y_{p.n} = (C_1x + C_2x^2 + C_3x^3)e^{kx}$, (c) $y_{p.n} = (C_1x^2 + C_2x^3 + C_3x^4)e^{kx}$, (d) $y_{p.n} = (C_1 + C_2x + C_3x^2)e^{kx}$, (e) $y_{p.n}(C_1x^2 + C_2x^3 + C_2x^4)e^{kx}$, (f) $y_{p.n} = (C_1x^3 + C_2x^4 + C_3x^5)e^{kx}$.

472. (a) $y_{p.n} = C_1 \sin x + C_2 \cos x$, (b) $y_{p.n} = x(C_1 \sin x + C_2 \cos x)$. **473.** (a) $y_{p.n} = x(C_1 \sin 2x + C_2 \cos 2x)e^{3x}$, (b) $y_{p.n} = x^2(C_1 \cos 2x + C_2 \sin 2x)e^{3x}$. **474.** $y_{p.n} = Cx$.

475. $y_{p.n} = C_1x + C_2x^2 + C_3x^3$. **476.** $y_{p.n} = Ce^x$.

477 $y_{p.n} = Cxe^{-7x}$. **478.** $y_{p.n} = (C_1x^2 + C_2x^3)e^{4x}$.

479. $y_{p.n} = Cx^2e^{5x}$. **480.** $y_{p.n} = (C_1x + C_2x^2)e^{\frac{3}{4}x}$.

481 $y_{p.n} = (C_1x + C_2x^2)e^{4x}$. **482.** $y_{p.n} = x(C_1 \cos 5x + C_2 \sin 5x)$. **483.** $y_{p.n} = x(C_1 \cos x + C_2 \sin x)$.

484. $y_{p.n} = x(C_1 \cos 4x + C_2 \sin 4x)$. **485.** $y_{p.n} = (C_1 \cos 2x + C_2 \sin 2x)e^{2x}$. **486.** $y_{p.n} = x(C_1 \cos 2x + C_2 \sin 2x)e^{2x}$.

487. $y_{p.n} = x(C_1 \cos 2x + C_2 \sin 2x)e^{-3x}$.

488. $y_{\text{р. п}} = x(C_1 \cos kx + C_2 \sin kx)$. 489. $y_{\text{р. п}} = C$ ($C = \text{const}$).
490. $y_{\text{р. п}} = C_1 + C_2 x$. 491. $y_{\text{р. п}} = C$ ($C = \text{const}$).
492. $y_{\text{р. п}} = Cx$. 493. $y_{\text{р. п}} = Cx^2$. 494. $y_{\text{р. п}} = C$ ($C = \text{const}$).
495. $y_{\text{р. п}} = Cx$. 496. $y_{\text{р. п}} = Cx^2$. 497. $y_{\text{р. п}} = Cx^3$.
498. $y_{\text{р. п}} = Cx^2$. 499. $y_{\text{р. п}} = Ce^{4x}$. 500. $y_{\text{р. п}} = Cx^2 e^{-x}$.
501. $y_{\text{р. п}} = (C_1 x^2 + C_2 x^3) e^{-x}$. 502. $y_{\text{р. п}} = C_1 \cos 2x + C_2 \sin 2x$. 503. $y_{\text{р. п}} = C_1 \cos x + C_2 \sin x$.
504. $y_{\text{р. п}} = (A_1 + A_2 x) \sin 2x + (B_1 + B_2 x) \cos 2x$.
505. $y_{\text{р. п}} = x^2 (C_1 \cos nx + C_2 \sin nx)$. 506. $y_{\text{р. п}} = C_1 \cos nx + C_2 \sin nx$. 507. $y_{\text{р. п}} = C_1 \sin x + C_2 \cos x$.
508. $y_{\text{р. п}} = Cx^4 e^x$. 509. $y_{\text{р. п}} = (C_1 x^4 + C_2 x^5) e^x$.
510. $y = (C_1 + C_2 x) e^{-x} - 2$. 511. $y = C_1 + C_2 e^{-2x} - x$.
512. $y = C_1 \cos 3x + C_2 \sin 3x + 1$. 513. $y = C_1 + C_2 x + C_3 e^{-x} + \frac{x^2}{2}$. 514. $y = C_1 + C_2 x + C_3 e^{\frac{7}{5}x} - \frac{3}{14} x^2$.
515. $y = C_1 + C_2 x + C_3 x^2 + C_4 e^{6x} + \frac{1}{6} x^3$.
516. $y = C_1 + C_2 x + C_3 x^2 + C_4 e^{-\frac{x}{3}} + \frac{x^3}{3}$.
517. $y = C_1 \cos x + C_2 \sin x + (C_3 + C_4 x) e^x + 1$.
518. $y = (C_1 + Cx) e^{2x} + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}$.
519. $y = C_1 + C_2 e^{-8x} + \frac{x^2}{2} - \frac{x}{8}$.
520. $y = (C_1 + C_2 x) e^{kx} + \frac{e^x}{(k-1)^2}$. 521. $y = (C_1 + C_2 x) e^{-2x} + 4x^2 e^{-2x}$. 522. $y = C_1 e^{-3x} + C_2 e^{-x} - \frac{9}{2} x e^{-3x}$.
523. $y = C_1 + C_2 e^{\frac{x}{7}} - 7x^2 - 98x$. 524. $y = C_1 + C_2 e^{-3x} - \left(\frac{x^2}{x} + \frac{x}{3}\right) e^{-3x}$. 525. $y = C_1 e^{-3x} + C_2 e^{-2x} + (20x - 5x^2) e^{-2x}$. 526. $y = (C_1 \cos x + C_2 \sin x) e^{-x} + \frac{x}{2}$.
527. $y = \left(C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x\right) e^{-\frac{x}{2}} + \frac{1}{3} (x^2 - x + 1) e^x$.
528. $y = C_1 e^{-(\sqrt{6}+2)x} + C_2 e^{(\sqrt{6}-2)x} - \frac{16 \cos 2x + 12 \sin 2x}{25}$.

529. $y = C_1 \cos x + C_2 \sin x + x \cos x + x^2 \sin x$.

530. $y = (C_1 + C_2 x)\, e^{mx} + \dfrac{2mn \cos nx + (m^2 - n^2) \sin nx}{(m^2 + n^2)^2}$.

531. $y = (C_1 \cos 2x + C_2 \sin 2x)\, e^{-x} - \dfrac{1}{4} x e^{-x} \cos 2x$.

532. $y = C_1 \cos ax + C_2 \sin ax + \dfrac{2 \cos mx + 3 \sin mx}{a^2 - m^2}$ $(|a| \neq |m|)$.

533. $y = C_1 + C_2 e^x - \dfrac{1}{2} (\cos x + \sin x)\, e^x$.

534. $y = C_1 + C_2 e^{-2x} + \dfrac{1}{5} (6 \sin x - 2 \cos x)\, e^x$.

535. $y = (C_1 \cos x + C_2 \sin x)\, e^{-2x} + 5x e^{-2x} \sin x$.

536. $y = C_1 + C_2 e^{-2x} - \left(\dfrac{x}{10} + \dfrac{1}{50}\right) \cos x + \left(\dfrac{7}{50} - \dfrac{x}{20}\right) \sin x$.

537. $y = C_1 e^{2x} + C_2 e^x - \left(\dfrac{x^2}{2} + x\right) e^x$.

538. $y = C_1 e^x + C_2 e^{-2x} + \dfrac{1}{18}\left(x^2 - x + \dfrac{7}{18}\right) e^{4x}$.

539. $y = C_1 e^x + C_2 e^{2x} + \left(\dfrac{x^2}{2} - x + 1\right) e^{3x}$.

540. $y = C_1 e^x + C_2 \cos x + C_3 \sin x - (x^2 + 3x + 1)$.

541. $y = (C_1 + C_2 x)\, e^x + C_3 \cos x + C_4 \sin x + \dfrac{x^2}{4} e^x$.

542. $y = (C_1 + C_2 x)\, e^x + x^3 + 6x^2 + 18x + 24$.

543. $y = C_1 + C_2 x + C_3 \cos x + C_4 \sin x + \dfrac{x^4}{12} + \dfrac{x^3}{6} - x^2$.

544. $y = \left(C_1 + \dfrac{x}{4} - \dfrac{x^3}{6}\right) \cos x + \left(C_2 + \dfrac{x^2}{4}\right) \sin x$.

545. $y = (C_1 + C_2 x)\, e^{-x} + [(6 - x^2) \cos x + 4x \sin x]\, e^{-x}$.

546. $y = C_1 e^x + \left(C_2 \cos \dfrac{\sqrt{3}}{2} x + C_3 \sin \dfrac{\sqrt{3}}{2} x\right) e^{-\frac{x}{2}}$
$+ \dfrac{1}{2} (\cos x - \sin x)$. 547. $y = (C_1 + C_2 x)\, e^x$
$+ (C_3 + C_4 x)\, e^{-x} + \dfrac{1}{4} \cos x$.

548. $y = (C_1 + C_2 x + C_3 x^2)\, e^x - \dfrac{e^x}{8} \sin 2x$.

549. $y = \left(C_1 \cos x + C_2 \sin x - \dfrac{x}{2} \cos x + x \sin x\right) e^{2x}$.

550. (a) $y = x (C_1 e^x + C_2 e^{-x})$, (b) $y = C_1 e^x + C_2 e^{-x}$,
(c) $y = x (C_1 e^x + C_2 x e^{-x})$, (d) $y = C_1 e^x + C_2 x^3 e^{-x}$.

551. $y_{\text{р.п}} = A_1 e^x + A_2 e^{-x} x$. 552. $y_{\text{р.п}} = x(A_1 x + A_2) + Bx e^{-4x}$.
553. $y_{\text{р.п}} = A_1 x + A_2 + B_1 \cos x + B_2 \sin x$.
554. $y_{\text{р.п}} = Ae^x + xe^x (B_1 \cos x + B_2 \sin x)$.
555. $y_{\text{р.п}} = Ax^2 + Bxe^x$. 556. $y_{\text{р.п}} = Ae^{2x} + x(B_1 \cos 2x + B_2 \sin 2x)$. 557. $y_{\text{р.п}} = A_1 \cos x + B_1 \sin x + A_2 \cos 3x + B_2 \sin 3x$. 558. $y_{\text{р.п}} = A_1 x + B_1 \cos 8x + B_2 \sin 8x$.
559. $y = C_1 e^{-x} + C_2 e^{2x} - 2x + 1 + e^x$.
560. $y = C_1 + C_2 e^{3x} - 3x^2 - 2x + \cos x + 3 \sin x$.
561. $y = 2 + e^x (C_1 + C_2 x - \sin x)$. 562. $y = (C_1 \cos x + C_2 \sin x) e^{-x} + xe^x + e^{-x}$. 563. $y = (C_1 \cos 2x + C_2 \sin 2x) e^{-x} + e^{-x} - 4 \cos 2x + \sin 2x$.
564. $y = C_1 e^{2x} + C_2 e^{-\frac{x}{2}} + \frac{1}{4}(2xe^{2x} - 5)$.
565. $y = C_1 \cos 2x + C_2 \sin 2x + \frac{x}{8}\left(1 + \frac{\cos 2x}{4} - \frac{x}{2} \sin 2x\right)$.
566. $y = (C_1 + C_2 x) e^{-x} + C_3 \cos x + C_4 \sin x - \frac{x}{8} \cos x + \frac{1}{4}\left(\frac{x}{2} - 1\right) e^x$. 567. $y = C_1 + C_2 e^{-x} + \frac{1}{2} e^x - \frac{1}{10} \cos 2x + \frac{1}{20} \sin 2x + \frac{x^3}{3} - x^2 + 2x$.
568. $y = C_1 + C_2 x + C_3 x^2 + C_4 \cos 2x + C_5 \sin 2x + \frac{e^x}{5} + \frac{x^3}{24} + \frac{3x \sin 2x}{32}$. 569. $y = (C_1 \cos 2x + C_2 \sin 2x) e^x + \cos x + 2 \sin x + 4 \cos 2x + \sin 2x$.
570. $y = C_1 + C_2 e^{-x} + xe^{-x} + \frac{1}{2} e^x + \frac{x^3}{3} - x^2 + 2x$.
571. $y = C_1 e^{-x} + C_2 e^{3x} - \frac{2}{3} x + \frac{4}{9} - \frac{1}{4} xe^{-x} - \frac{1}{2} xe^{3x}$.
572. $y = C_1 \cos 2x + C_2 \sin 2x + x\left(\frac{1}{4} \sin 2x - \cos 2x\right) + \frac{1}{5} e^x$.
573. $y = C_1 e^{-x} + C_2 e^{-2x} + 3(x^2 - 2x) e^{-x} + 3(x^2 + 2x) e^{-2x}$.
574. $y = C_1 \cos x + C_2 \sin x - \frac{1}{3} \cos 4x - \frac{x}{4} \sin x + 1$.
575. $y = (C_1 \cos x + C_2 \sin x) e^{2x} + \cos x - \sin x + e^{2x} + \frac{1}{5}$.
576. $y = \left(C_1 \cos x + C_2 \sin x + \frac{1}{2} - \frac{x}{4} \sin x\right) e^x$.

577. $y=C_1+C_2e^{3x}+\frac{\cos x-2\sin x}{5}-\frac{e^x}{2}-\frac{x}{3}$.

578. $y=\left(C_1\cos 2x+C_2\sin 2x+\frac{x}{4}\sin 2x\right)e^x+2x+1$.

579. $y=(C_1+C_2x)\,e^x+\frac{1}{25}(4\cos x+3\sin x)$
$+\frac{1}{8}\cos 2x+x+1$.

580. $y=(C_1+C_2x)\,e^{-x}+\frac{\cos 2x+7\sin 2x}{25}+\sin x+1$.

581. $y=\left(C_1\cos\frac{\sqrt{3}}{2}x+C_2\sin\frac{\sqrt{3}}{2}x\right)e^{-\frac{x}{2}}-\cos x+x^2$
$-x-2$. 582. $y=(C_1+C_2x+9x^2)\,e^{-3x}+\sin x$.

583. $y=C_1+C_2e^{-2x}-\frac{\cos x}{5}+\frac{2\sin x}{5}$
$-\frac{3}{8}(\cos 2x+\sin 2x)-\frac{x}{2}$.

584. $y=C_1+(C_2+C_3x)\,e^x+x^2+4x+\frac{1}{2}x^2e^x$.

585. $y=C_1\cos x+C_2\sin x+\frac{1}{8}\cos 3x+\frac{1}{2}x\sin x$.

586. $y=C_1+C_2e^{-x}+C_3e^{2x}+x-x^2+\cos x$.

587. $y=C_1+C_2e^{2x}+C_3e^{-2x}+\frac{1}{5}\cos x-\frac{x^3}{12}-\frac{x}{8}$
$+\frac{e^{3x}}{32}(2x^2-3x)$. 588. $y=C_1x^3+C_2x^2+C_3x+C_4$
$+C_5e^x+\frac{x^4}{24}\left(\frac{x^2}{3}-4x\right)e^x$. 589. $y=C_1+C_2x$
$+C_3x^2+C_4\cos x+C_5\sin x+\frac{x^4}{24}-e^{-x}$.

590. $y=2-2x$. 591. $y=x^2+e^{3x}$. 592. $y=2e^{3x}$.

593. $y=x^2e^{2x}$. 594. $y=e^{2x}-e^{3x}+xe^{-x}$.

595. $y=1-xe^{-x}$. 596. $y=\left(x+\frac{3}{5}\right)e^{-3x}$
$+\frac{1}{5}(4\sin x-3\cos x)$. 597. $y=\cos x+x\sin x$.

598. $y=\cos 2x+\frac{1}{3}(\sin 2x+\sin x)$. 599. $y=x\cos x$
$+x^2\sin x$. 600. $y=(\cos x-2\sin x)\,e^{2x}+(x-1)^2e^x$.

601. $y=xe^{3x}+x+e^{-x}$. 602. $y=2e^x+(\sin x-2\cos x)\,e^{-x}-4$.

603. $y=-[\pi\cos x+(\pi+1-2x)\sin x]\,e^x$.

604. $y=\sinh x+x^2$. 605. $y=\cos x+2\sin x$
$+e^{-x}+(2x-3)e^x$.

606. $y=2x-\frac{4}{\sqrt{3}}e^{-\frac{x}{2}}\sin\frac{\sqrt{3}}{2}x$. 607. $y=2xe^x$.

608. $y=\frac{1}{4}\cos x$. 609. $y=\sin 2x$. 610. $y=-1$.

611. $y=\cos x$. 612. $y=e^{-x}$. 613. $y=e^x+3$.

614. $y=-\frac{1}{5}$. 615. $y=(\cos x+\sin x)e^x$

616. $y=e^{-2x}\cos 2x$. 617. $y=(x^2+x)e^{-x}$.

618. $y=C_1x+\frac{C_2}{x}$. 619. $y=\frac{1}{x}(C_1+C_2\ln x)$.

620. $y=\frac{1}{\sqrt{x}}\left[C_1\cos\left(\frac{\sqrt{23}}{2}\ln x\right)+C_2\sin\left(\frac{\sqrt{23}}{2}\ln x\right)\right]$.

621. $y=C_1+C_2\ln x$. 622. $y=C_1(x-2)+C_2(x-2)^{-3}$.

623. $y=C_1(2x+1)+C_2(2x+1)\ln(2x+1)$. 624. $y=C_1$
$+C_2x^2+C_3x^4$. 625. $y=C_1+C_2x^3+C_3\ln x$.

626. $y=C_1+C_2(x+1)^5+C_3(x+1)^{-2}$. 627. $y=C_1$
$+(2x+1)\left[C_2\cos\frac{\ln(2x+1)}{\sqrt{2}}+C_3\sin\frac{\ln(2x+1)}{\sqrt{2}}\right]$.

628. $y=C_1\cos\ln x+C_2\sin\ln x+\frac{x}{2}(7-\ln x)$.

629. $y=C_1x^2+\frac{C_2}{x}+\frac{1}{10}(\cos\ln x-3\sin\ln x)$.

630. $y=\frac{1}{x}(C_1+C_2x^4+\ln x+2\ln^2 x)$. 631. $y=C_1x$
$+C_2x^2+(x^2+2x)\ln x+1$.

632. $y=C_1x+\frac{C_2}{x}+\frac{x^m}{m^2-1}$, $|m|\neq 1$.

633. $y=\frac{C_1}{x}+\frac{C_2}{x}+\ln^2 x-3\ln x+2x+7$.

634. $y=\frac{1}{x+1}[C_1+C_2\ln(x+1)+\ln^3(x+1)]$.

635. $y=(x-2)^2[C_1+C_2\ln(x-2)]+x-\frac{3}{2}$.

636. $y=C_1(1+4x^2)+C_2e^{-2x}$. 637. $y=C_1(2x-3)+C_2x^{-2}$.

638. $y=C_1x^3+C_2(x+1)-x$. 639. $y=C_1x+C_2\ln x$.

640. $y=C_1\sin x+C_2\sin^2 x$. 641. $y=C_1\cos(\sin x)$
$+C_2\sin(\sin x)$. 642. $y=C_1x+C_2\sqrt{1+x^2}+1$.

643. $y=\frac{C_1}{x}+C_2x^3+x^4$. 644. $y=C_1x+\left(C_2-x+\frac{x^2}{2}\right)e^x$.

645. $y=C_1\cos(e^{-x})+C_2\sin(e^{-x})+e^{-x}$.

646. $y=\frac{C_1}{x}+C_2e^{1/x}-\frac{\ln|x|}{x}+1$. 647. $y=C_1\cos e^x$
$+C_2\sin e^x+x$. 648. $y=C_1(2x-1)+C_2x^2+x^3$.

649. $\frac{d^2x}{dt^2}=\frac{k}{m}x$ (x being the length of the hanging portion of the chain);

$$t=\sqrt{\frac{6}{g}}\ln(6+\sqrt{35})\text{ s. } k=g,\ m=6.$$

650. $\frac{d^2S}{dt^2}=1.2t,\ S=0.2t^3-t$. 651. $m\frac{d^2S}{dt}=-km$,
$S=\frac{v_0^2}{2k}$. 652. $\frac{d^2x}{dt^2}=k^2x,\ x=ae^{kt}$.

653. $y=(C_1-x)\cos x+(C_2+\ln|\sin x|)\sin x$.

654. $y=C_1e^x+C_2+(e^x+1)\ln(1+e^{-x})$.

655. $y=C_1\cos x+C_2\sin x-\frac{\cos 2x}{2\cos x}$. 656. $y=C_1\cos x$
$+C_2\sin x+\frac{4}{3}\cos x\sqrt{\cot x}$. 657. $y=(C_1+C_2x)e^x$
$-e^x\ln\sqrt{1+x^2}+e^x x\arctan x$.

658. $y=(C_1-x)e^{-x}\cos x+(C_2+\ln|\sin x|)e^{-x}\sin x$.

659. $y=C_1\cos x+C_2\sin x+\frac{\cos 2x}{\sin x}$. 660. $y=C_1e^x+C_2$
$-\cos e^x$. 661. $y=C_1+C_2x+C_3e^{-x}+1-x+x\ln|x|$.

662. $y=C_1e^{x^2}+C_2+(x^2-1)e^{x^2}$. 663. $y=C_1+C_2\tan x$
$+\frac{1}{2}(1+x\tan x)$. 664. $y=C_1x(\ln x-1)+C_2$
$+x(\ln^2 x-2\ln x-2)$. 665. $y=C_1\left(x+\frac{1}{2}\right)e^{-2x}+C_2$
$-x^2$. 666. $y=C_1\sin x+C_2+(\ln|\sin x|-1)\sin x$.

667. $y=1$.

668. $y=\frac{1}{x}$. 669. $y=\frac{1}{2}\arctan^2 x$. 670. $y=\left(1+x-\frac{x^2}{2}\right)e^x$. 671. $y=\frac{1-\ln x}{\sqrt{x}}$. 672. $y=(x-1)e^x$.

673. $y=\frac{1}{x}$. 674. $y=1$. 675. $y''-y=0$.

676. $(x-1)y''-xy'+y=0$. 677. $(x-1)y''-x^2y'+(x^2-x+1)y=0$. 678. $y'''=0$. 679. $xy'''-y''+xy'-y=0$. 681. $y=C_1y_1+C_2y_1\int\frac{1}{y_1^2}e^{\int-p_1\,dx}\,dx$.

682. $p_0(x)=W(x)$, $p_1(x)=-W'(x)$, $p_2(x)=W(y_1', y_2')$, where $W(x)\equiv W(y_1, y_2)$ is the Wronskian.

685. $p_1^2<4p_2$. 689. $v(x)=e^{-\frac{1}{2}\int p(x)\,dx}$. 691. $p>0$, $q>0$. 692. $p=0$, $q>0$.

693. Assume that $y(x)>0$ in (a, b). Since the solution $y(x)$ satisfies all the conditions of the Rolle theorem, there must exist a point $\xi\in(a, b)$ in which $y'(\xi)=0$ and so

$$y''(\xi)=\frac{\xi^2+4}{\xi^2+1}>0,$$

a contradiction, for at the point $x=\xi$ the solution $y(x)$ has a maximum and $y''(\xi)<0$ must hold. Therefore $y(\xi)<0$. Similarly it is possible to show that we shall have $y(\xi)<0$ at the other points ξ as well, if there are any. It follows that $y(x)<0$ in (a, b).

706. (a) $\lambda=k^2$, $k=0, 1, 2, \ldots$; (b) $\lambda=4k^2$, $k=0, 1, 2, \ldots$.

707. For any λ. 708. (a) is solvable, $y=\frac{\sinh x}{\sinh 2\pi}$; (b) is unsolvable. 709. (1) $\lambda-\omega^2>0$, $y=C_1\cos 2n\pi x+C_2\sin 2n\pi x$; (2) $\lambda-\omega^2=0$, $y\equiv C=\text{const}$; (3) $\lambda-\omega^2<0$, $y\equiv 0$. 710. $y=\sqrt{1+4x-x^2}$.

771. $y=\alpha\sin x$. 712. $y=\frac{\sinh x}{\cosh 1}$. 713. $y=-e^x\sin x$.

714. $y=e^\alpha$. 715. $y=\frac{1}{\alpha^2}+\frac{\cos\alpha(\pi-x)}{\sin\alpha\pi}$, $0<\alpha<1$.

716. $y=1-\cos x$. 717. $\lambda=n$, $y=\cos nx$, $n=1, 2, \ldots$.

718. $\lambda=n+\frac{1}{2}$, $y=\sin\left(n+\frac{1}{2}\right)x$, $n=0, 1, \ldots$.

719. $y=(x-1)e^{-x}$. 720. $y=C\sin nx$, $n=0, 1, 2, \ldots$, C being an arbitrary constant. 721. $y\equiv 0$.

722. $y=C(x\ln x-x+1)$, C being an arbitrary constant.

723. $y\equiv 0$. 724. $y=x-\frac{x^3}{3}+\frac{x^5}{3\times 5}-\ldots$. 725. $y=1+x-x^2+\ldots$. 726. $y=1+\frac{x^2}{2}+\frac{x^3}{12}+\ldots$.

727. $y=x-\frac{2x^4}{4!}+\frac{10x^7}{7!}-\ldots$. 728. $y=x+\frac{x^3}{3!}+\frac{2x^5}{5!}+\ldots$. 729. $y=1+\frac{(x-\pi)^2}{2}+\frac{(x-\pi)^3}{3\pi}+\ldots$.

730. $y=\frac{1}{e}+\frac{\sin 1}{2!}(x-e)^2+\frac{\cos 1-\sin 1}{3!\,e}(x-e)^3+\ldots$.

731. $y=\frac{\pi}{2}-\frac{x^4}{4!}+\frac{2x^6}{6!}-\ldots$. 732. $y=1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\ldots$; $(=e^{x^2})$. 733. $y=C_1\left[1-\frac{x^2}{2}+\frac{x^4}{2\times 4}-\ldots+\frac{(-1)^n x^{2n}}{2\times 4\ldots 2n}+\ldots\right]+C_2\left[x-\frac{x^3}{1\times 3}+\frac{x^5}{1\times 3\times 5}-\ldots+\frac{(-1)^n x^{2n+1}}{1\times 3\times 5\ldots(2n+1)}+\ldots\right]$.

734. $y=\frac{x^2}{2}+\frac{x^4}{4!}+\frac{3x^6}{6!}+\frac{3\times 5x^8}{8!}+\ldots+\frac{(2n+1)!!\,x^{2n+4}}{(2n+4)!}+\ldots$, where $(2n+1)!!=1\times 3\times 5\ldots(2n+1)$.

735. $y=-2+2x-x^2+\frac{x^3}{3}-\frac{x^4}{4}+\frac{7x^5}{60}-\ldots$. 736. $y=1+\frac{2x^4}{4!}-\frac{2x^5}{5!}+\frac{2x^6}{6!}-\frac{2x^7}{7!}+\frac{62x^8}{8!}-\ldots$.

737. $y=C_1\left(1+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{12}+\frac{x^5}{24}+\frac{13x^6}{720}+\ldots\right)+C_2\left(x+\frac{x^3}{6}+\frac{x^4}{12}+\frac{x^5}{30}+\frac{x^6}{72}+\frac{29x^7}{5040}+\ldots\right)$.

738. $y=x+\frac{x^2}{2}+\frac{2x^3}{3}+\frac{11x^4}{24}+\frac{53x^5}{120}+\frac{269x^6}{720}+\ldots$.

739. $y=C_1\left(1-\frac{x}{2!}+\frac{x^2}{4!}-\ldots\right)+C_2\sqrt{x}\left(1-\frac{x}{3!}+\frac{x^2}{5!}-\ldots\right)$; $(y=C_1\cos\sqrt{x}+C_2\sin\sqrt{x})$.

740. $y=C\sum_{k=0}^{\infty}\frac{n(n-1)\dots(n-k+1)}{k!}x^k.$

741. $y=C_1\left(1+\frac{x}{3}+\frac{1\times 4x^2}{3\times 6}+\frac{1\times 4\times 7x^3}{3\times 6\times 9}+\dots\right)$
$+C_2x^{\frac{7}{3}}\left(1+\frac{8x}{10}+\frac{8\times 11x^2}{10\times 13}+\frac{8\times 11\times 14x^3}{10\times 13\times 16}+\dots\right).$

742. $y=C_0x^{\frac{m}{2}}\left\{1-\frac{\alpha x}{m+1}+\left[\frac{\alpha^2}{2(m+1)(m+2)}-\frac{E}{4(m+2)}\right]\right.$
$\left.\times x^2+\dots\right\}$, C_0 being an arbitrary constant, $\rho=\frac{m}{2}$.

743. $y=C_0x\left\{1+\frac{2-\alpha}{6}x^2+\left[\frac{(2-\alpha)(12-\alpha)}{120}-\frac{\beta}{20}\right]x^4+\dots\right\}$,
C_0 being an arbitrary constant, $\rho=1$. 744. $y=C_1J_{1/3}(2x)$
$+C_2J_{-1/3}(2x)$. 745. $y=C_1J_{1/2}(x)+C_2J_{-1/2}(x)$.

746. $y=C_1J_0\left(\frac{x}{3}\right)+C_2Y_0\left(\frac{x}{3}\right)$. 747. $y=C_1J_0(2x)$
$+C_2Y_0(2x)$. 748. $y=x^{3/2}[C_1J_{5/4}(x^2)+C_2J_{-5/4}(x^2)]$.

749. $y=\sqrt[4]{x}\,[C_1J_{1/2}(\sqrt{x})+C_2J_{-1/2}(\sqrt{x})]$.

750. $y=\frac{1}{x^2}[C_1J_2(x)+C_2Y_2(x)]$.

751. $y=\frac{1}{x}[C_1J_1(2x)+C_2Y_1(2x)]$.

752. $y=\frac{1}{3}-\sum_{n=1}^{\infty}\frac{\cos nx+\sin nx}{n^3(n^2-3)}$. 753. No periodic solutions. 754. $y=C_1\cos x+C_2\sin x-\sum_{n=2}^{\infty}\frac{\cos nx}{n^2(n^2-1)}$.

755. No periodic solutions.

756. $y=C-\sum_{n=1}^{\infty}\frac{\cos nx+n\sin nx}{n^3(n^2+1)}$.

757. No periodic solutions. 758. $y=\frac{\pi^2}{6}$
$+4\sum_{n=1}^{\infty}(-1)^n\frac{(n^2-4)\cos nx+4n\sin nx}{n^2(n^2+4)^2}$.

759. $y=-\frac{1}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{\cos 2n\pi x}{(n^2\pi^2+1)(4n^2-1)}$.

760. $y=\frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{4n\cos nx+(4-n^2)\sin nx}{(2n-1)^2(n^2+4)^2}$.

761. No periodic solutions. 764. $f(x)\sim\frac{1}{x}$. 767. Yes.
768. Yes. 769. No. 770. No. 771. Yes. 772. No.
773. (a) Yes; (b) No. 774. Yes. 775. No.

776. $\begin{cases} x=3C_1\cos 3t-3C_2\sin 3t, \\ y=C_2\cos 3t+C_1\sin 3t. \end{cases}$

777. $\begin{cases} x=C_1e^t-C_2e^{-t}+t-1, \\ y=C_1e^t+C_2e^{-t}-t+1. \end{cases}$

778. $\begin{cases} x=-2e^{-t}+3e^{-7t}, \\ y=e^{-t}+3e^{-7t}. \end{cases}$

779. $\begin{cases} x=(\sin t-2\cos t)\,e^{-t}, \\ y=e^{-t}\cos t. \end{cases}$

780. $\begin{cases} x=C_1e^{-t}+C_2e^{-3t}, \\ y=C_1e^{-t}+3C_2e^{-3t}+\cos t. \end{cases}$

781. $\begin{cases} x=(C_1-C_2)\cos t+(C_1+C_2)\sin t, \\ y=C_1\sin t-C_2\cos t+C_3e^t, \\ z=C_1\cos t+C_2\sin t+C_3e^t. \end{cases}$

782. $\begin{cases} x=C_1e^{-t}+C_2e^{2t}, \\ y=C_3e^{-t}+C_2e^{2t}, \\ z=-(C_1+C_3)\,e^{-t}+C_2e^{2t}. \end{cases}$

783. $\begin{cases} x=C_1e^t+C_2e^{-t}+C_3\sin t+C_4\cos t, \\ y=C_1e^t+C_2e^{-t}-C_3\sin t-C_4\cos t. \end{cases}$

784. $\begin{cases} x=C_1+C_2t+C_3t^2, \\ y=-(C_1+2C_3)\,t-\frac{C_2}{2}t^2-C_3\frac{t^3}{3}+C_4. \end{cases}$

785. $\begin{cases} x=(C_1+C_2t)\,e^t+C_3e^{-2t}, \\ y=2\,(C_2-C_1-C_2t)\,e^t+C_3e^{-2t}. \end{cases}$

786. $\begin{cases} x = e^t, \\ y = e^t - e^{2t}. \end{cases}$ 787. $\begin{cases} \dfrac{1}{x+y} + t = C_1, \\ \dfrac{1}{x-y} + t = C_2. \end{cases}$

788. $\begin{cases} x = C_2 e^{-t/C_1}, \\ y = \dfrac{C_1}{C_2} e^{t/C_1}. \end{cases}$ 789. $\begin{cases} \dfrac{1}{x} - \dfrac{1}{y} = C_1, \\ 1 + C_1 x = C_2 e^{C_1 t}. \end{cases}$

790. $\begin{cases} x^2 - y^2 = C_1, \\ x - y + t = C_2. \end{cases}$ 791. $\begin{cases} \tan \dfrac{x+y}{2} = C_1 e^t, \\ \tan \dfrac{x-y}{2} = C_2 e^t. \end{cases}$

792. $\begin{cases} y = C_1 x, \\ C_1 x^2 = C_2 - 2e^{-t}. \end{cases}$ 793. $\begin{cases} \tan(x+y) = t, \\ \tan(x-y) = t. \end{cases}$

794. $\begin{cases} x = C_1 t, \\ y = C_2 e^t. \end{cases}$ 795. $\begin{cases} t^2 - x^2 = C_1, \\ x^2 - y^2 = C_2. \end{cases}$

796. $\begin{cases} x^2 + y^2 = C_1^2, \\ p^2 + q^2 = C_2^2, \\ xp + yq = C_3. \end{cases}$ 797. $\begin{cases} xy = C_1, \\ \ln x = C_2 + \dfrac{t^2}{2C_1}. \end{cases}$

798. $\begin{cases} x^2 + y^2 = C_1 x - t^2, \\ y = C_2 x. \end{cases}$ 799. $\begin{cases} 2x + 3y + 4t = C_1, \\ x^2 + y^2 + t^2 = C_2. \end{cases}$

800. $\begin{cases} x = \dfrac{t}{3} + \dfrac{C_2}{t^2}, \\ y = C_1 e^t - \dfrac{t}{3} - \dfrac{C_2}{t^2}. \end{cases}$ 801. $\begin{cases} x^2 + y^2 + t^2 = C_1, \\ x^2 - 2xy - y^2 = C_2. \end{cases}$

802. $\begin{cases} x = 2C_1 e^{3t} - 4C_2 e^{-3t}, \\ y = C_1 e^{3t} + C_2 e^{-3t}. \end{cases}$ 803. $\begin{cases} x = C_1 + C_2 e^t, \\ y = C_1. \end{cases}$

804. $\begin{cases} x \equiv 0, \\ y \equiv 0. \end{cases}$ 805. $\begin{cases} x = e^{2t} - e^{3t}, \\ y = e^{2t} - 2e^{3t}. \end{cases}$

806. $\begin{cases} x = -5e^{2t} \sin t, \\ y = e^{2t}(\cos t - 2 \sin t). \end{cases}$

807. $\begin{cases} x = \dfrac{1}{3} C_1 e^t - C_2 e^{-2t}, \\ y = \dfrac{1}{3} C_1 e^t + 2C_2 e^{-2t}, \\ z = \dfrac{1}{3} C_1 e^t - C_2 e^{-2t}. \end{cases}$

808. $\begin{cases} x = C_1 e^{2t} - C_2 e^{3t}, \\ y = C_1 e^{2t} - C_3 e^{t}, \\ z = C_1 e^{2t} - C_2 e^{3t} - C_3 e^{t}. \end{cases}$

809. $\begin{cases} x = 1 - e^{-t}, \\ y = 1 - e^{-t}, \\ z = 2e^{-t} - 1. \end{cases}$

810. $\begin{cases} x = \frac{8}{3} e^{2t} + 2C_1 e^{t} + C_3 e^{-t}, \\ y = \frac{29}{3} e^{2t} + 3C_1 e^{t} + C_2 e^{-t}. \end{cases}$

811. $\begin{cases} x = (1 - t)\cos t - \sin t, \\ y = (t - 2)\cos t + t \sin t. \end{cases}$

812 $\begin{cases} x = C_1 \cos t + C_2 \sin t + \tan t, \\ y = -C_1 \sin t + C_2 \cos t + 2. \end{cases}$

813. $\begin{cases} x = C_1 + 2C_2 e^{-t} + 2e^{-t} \ln |e^{t} - 1|, \\ y = -2C_1 - 3C_2 e^{-t} - 3e^{-t} \ln |e^{t} - 1|. \end{cases}$

814. $\begin{cases} x = C_1 \cos t + C_2 \sin t + \cos t \ln |\cos t| + t \sin t, \\ y = -C_1 \sin t + C_2 \cos t - \sin t \ln |\cos t| + t \cos t. \end{cases}$

815. $\begin{cases} x = C_1 \cos t + C_2 \sin t + 1, \\ y = -C_1 \sin t + C_2 \cos t. \end{cases}$

816. $\begin{cases} x = C_1 \cos 2t + C_2 \sin 2t + t, \\ y = C_1 \sin 2t - C_2 \cos 2t + 1. \end{cases}$

817. $\begin{cases} x = -C_1 \sin t + (C_2 - 1)\cos t, \\ y = C_1 \cos t + C_2 \sin t. \end{cases}$

818. $\begin{cases} x = C_1 e^{2t} + C_2 + e^{t}, \\ y = C_1 e^{2t} - C_2 - e^{t}. \end{cases}$ 819. $\begin{cases} x = -t, \\ y = 0. \end{cases}$

820. $\begin{cases} x = e^{t}, \\ y = e^{t}. \end{cases}$

821. $\begin{cases} x = -C_1 \sin t + C_2 \cos t + t, \\ y = C_1 \cos t + C_2 \sin t + t^2 - 2. \end{cases}$

822. $\begin{cases} x = C_1 t + C_2 - 2e^{-t} - \cos t - \sin t, \\ y = C_1 - 2e^{-t} + \cos t. \end{cases}$

823. $\begin{cases} x = C_1 e^t + C_2 \sin t + C_3 \cos t, \\ y = -C_1 e^t + C_2 \cos t - C_3 \sin t + t, \\ z = C_2 \sin t + C_3 \cos t + 1. \end{cases}$

824. $\begin{cases} x = e^{-t}, \\ y = e^{-t}, \\ z = 1. \end{cases}$ 825. $\begin{cases} x = 4C_1 e^{6t} - C_2 e^t, \\ y = C_1 e^{6t} + C_2 e^t, \end{cases}$

826. $\begin{cases} x = C_1 e^{2t} + 4C_2 e^{7t}, \\ y = -4C_1 e^{2t} + 4C_2 e^{7t}. \end{cases}$

827. $\begin{cases} x = 4C_1 e^t + C_2 e^{6t} - \frac{5}{6}, \\ y = C_1 e^t - C_2 e^{6t} - \frac{1}{6}. \end{cases}$

828. $\begin{cases} x = C_1 e^{4t} + C_2 e^{2t} - e^t, \\ y = C_1 e^{4t} - C_2 e^{2t} + e^t. \end{cases}$

829. $\begin{cases} x = C_1 (1 + 2t) - 2C_2 - 2 \cos t - 3 \sin t, \\ y = -C_1 t + C_2 + 2 \sin t. \end{cases}$

830. $x = e^{-2t} - e^{-3t}$. 831. $x = -(t^3 + 2t^2 + 2t + 1)$.

832. $x = \sin t$. 833. $x = \frac{t^2 - 2}{4} e^{-3t}$.

834. $x = e^{-t} + \sin t - \cos t$. 835. $x \equiv 0$.

836. $x = \frac{1}{2} t^2$. 837. $x = 1 - \cos t$. 838. $x \equiv 0$.

839. $x = e^{-t}$. 840. $x = -1 = t$. 841. $x = t$.

842. $x = t^2$. 843. $x \equiv 1$. 844. $x = 1 - 4t e^{-2t}$.

845. $x = \frac{t^2 + 2}{4} e^t$. 846. $x = (t + 1) \sin t - \cos t$.

847. $x = t^2 - 3t + 4$. 848. $x = e^t + \sin t$.

849. $x = \left(\frac{t^3}{6} - \frac{t^2}{2} + t\right) e^{2t}$. 850. $x = \left(\frac{t^2}{8} + t - 2\right) e^{t/2}$.

851. $x = (1 + t) e^{-t} + (1 - t) e^{-2t}$.

852. $x = \frac{t}{5} (6e^{3t} - 2e^{-2t})$. 853. $x = \frac{t^4}{12} e^{-2t}$.

854. $x = e^t + \cos t - \sin t$. 855. $x = 3(1 + t) \sin 3t$.

856. $x = t\left(\sin 2t + \frac{1}{8} \cos 2t\right)$.

857. $x=\frac{1}{4}(t-1)(\cos t+\sin t)$.

858. $x=e^{2t}[(1-t)\cos t+(1+t)\sin t]$.

859. $x=t-1+2e^{t}$. 860. $x=-t/4$. 861. $x=te^{t}$.

862. $x=4t(\sin t-\cos t)$. 863. $x=\frac{1}{4}(1-\cos 2t+t\sin 2t)$.

864. $\begin{cases} x=e^{t}+e^{-t}, \\ y=-e^{t}+e^{-t}. \end{cases}$ 865. $\begin{cases} x=4e^{-2t}-3e^{-3t}, \\ y=3e^{-3t}-2e^{-2t}. \end{cases}$

866. $\begin{cases} x=e^{t}(\cos t-2\sin t), \\ y=e^{t}(3\sin t+\cos t). \end{cases}$

867. $\begin{cases} x=-\frac{5}{4}+\frac{13}{4}\cos 2t-3\sin 2t, \\ y=\frac{3}{2}t+3\cos 2t+\frac{13}{4}\sin 2t. \end{cases}$

868. $\begin{cases} x=e^{t}, \\ y=e^{t}. \end{cases}$ 869. $\begin{cases} x=\frac{1}{2}(\sin t+\cos t), \\ y=\frac{1}{2}(\sin t-\cos t). \end{cases}$

870. $\begin{cases} x=2-e^{t}, \\ y=-2+4e^{t}-te^{t}, \\ z=-2+5e^{t}-te^{t}. \end{cases}$ 871. $\begin{cases} x=1+3e^{2t}+e^{-2t}, \\ y=e^{2t}-e^{-2t}, \\ z=2e^{2t}+2e^{-2t}. \end{cases}$

872. $\begin{cases} x=3-2e^{-t}, \\ y=e^{-t}, \\ z=e^{-t}-3. \end{cases}$

873. $\begin{cases} x=2(1-e^{-t}-te^{-t}), \\ y=2-t-2e^{-t}-2te^{-t}. \end{cases}$

874. $\begin{cases} x=e^{t}+\sin t, \\ y=e^{t}-\sin t. \end{cases}$ 875. $\begin{cases} x=\cos t+e^{-\sqrt{3}\,t}, \\ y=\frac{1}{2}(\cos t-e^{-\sqrt{3}\,t}. \end{cases}$

876. $\begin{cases} x=t-\frac{t^{3}}{6}+e^{t}, \\ y=1+\frac{1}{24}t^{4}-e^{t}. \end{cases}$

877. $\begin{cases} x=12(\cosh t-1)-\frac{7}{2}t\times\sinh t. \\ y=7t\times\sinh t-17(\cosh t-1). \end{cases}$

878. $\begin{cases} x = t^2 + t, \\ y = -\frac{1}{2} t^2. \end{cases}$ 879. $\begin{cases} x = e^t (2\cos t - \sin t), \\ y = e^t (3\cos t + \sin t). \end{cases}$

880. The solution is unstable. **881.** The solution is unstable. **882.** The solution is stable. **883.** The solution is stable. **884.** The solution is stable. **885.** The solution is stable. **886.** The stationary point is an unstable focus. **887.** The stationary point is a saddle point. **888.** The stationary point is a midpoint. **889.** The stationary point is a stable focus. **890.** The stationary point is a stable node. **891.** The stationary point is an unstable node. **892.** The stationary point is an unstable node (a dicritical node). **893.** $\alpha < -3/2$. **894.** (a) The stationary point is asymptotically stable; (b) the stationary point is unstable; (c) the stationary point is asymptotically stable. **895.** $V = 2x^2 + 3y^2$, the trivial solution is asymptotically stable. **896.** $V = x^4 + y^4$, the trivial solution is stable. **897.** $V = x^2 - \frac{1}{2} y^2$, the trivial solution is unstable. **898.** $V = x^2 + y^2$, the trivial solution is asymptotically stable. **899.** $V = x^2 + y^2$, the trivial solution is unstable. **900.** $V = 2x^2 + y^2$, the trivial solution is stable. **901.** $V = x^2 + y^2$, the trivial solution is asymptotically stable. **902.** $V = x^2 - y^2$, the trivial solution is unstable. **903.** $V = x^2 + \frac{1}{2} y^2$, the trivial solution is stable. **904.** $V = x^2 + 3y^2$, the trivial solution is asymptotically stable. **905.** $V = x^4 - y^4$, the trivial solution is unstable. **906.** $V = x^4 + 2y^2$, the trivial solution is stable. **907.** $V = x^2 - y^2$, the trivial solution is unstable. **908.** $V = 3x^2 + 2y^2$, the trivial solution is stable. **909.** It is obvious from the conditions imposed on the function $v(x_1, \ldots, x_n)$ that the given system has the trivial solution $x_1 = 0, \ldots, x_n = 0$. Taking v as a Lyapunov function, the derivative $\frac{dv}{dt}$ is, by virtue of the system,

$$\frac{dv}{dt} = -\left[\left(\frac{\partial v}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial v}{\partial x_n}\right)^2\right] \ldots .$$

Hence the trivial solution $x = 0$ of the given system is a Lyapunov stable solution for $t \to +\infty$. The stability is asymptotic if $\frac{dv}{dt}$ is a negative definite function. **910.** The zero solution is unstable. **911.** The zero solution is stable.

912. The zero solution is stable. **913.** The zero solution is stable. **914.** The zero solution is stable. **915.** The zero solution is unstable. **916.** The zero solution is stable. **917.** The zero solution is unstable. **918.** The zero solution is stable. **919.** The zero solution is stable. **920.** The zero solution is unstable. **921.** No investigation in the first approximation is possible, since the roots of the characteristic equation are pure imaginary. There exists, however, a function $V = 3x^2 + 4y^2$ satisfying all the conditions of Lyapunov's asymptotic-stability theorem, in particular $\frac{dV}{dt} = -(6x^4 + 8y^4) \leqslant \leqslant 0$. Hence the stationary point $x = 0$, $y = 0$ is asymptotically stable. **922.** $V = x^2 + y^2$; the solution $x \equiv 0$, $y \equiv 0$ is asymptotically stable. **923.** $\Delta < 0.017$. **924.** $\Delta < 0.16$. **925.** $\Delta < 0.0012$. **926.** The zero solution is unstable. **927.** The zero solution is stable. **928.** The zero solution is stable. **929.** The zero solution is unstable. **930.** The zero solution is stable. **931.** The zero solution is unstable. **932.** The zero solution is stable. **933.** The zero solution is stable. **934.** The zero solution is stable. **935.** The zero solution is unstable. **936.** The zero solution is stable. **937.** $\alpha > \frac{3}{2}$. **938.** It is always unstable. **939.** $\alpha > \frac{5}{2}$. **940.** $\alpha > 0$, $\alpha\beta > 1 + \alpha^2$. **941.** $\alpha > \frac{2}{3}$, $\beta > 0$, $9\beta - 6\alpha + 4 < 0$. **942.** The zero solution is stable. **943.** The zero solution is stable. **944.** The zero solution is stable. **945.** The zero solution is stable. **946.** The zero solution is stable. **947.** The zero solution is stable. **948.** The zero solution is stable. **949.** The zero solution is unstable. **950.** The zero solution is unstable. **951.** The zero solution is unstable. **952.** The zero solution is unstable. **953.** The zero solution is stable. **954.** The zero solution is unstable. **955.** The zero solution is stable. **956.** The zero solution is stable. **957.** The zero solution is unstable. **958.** The zero solution is unstable. **959.** The zero solution is unstable. **960.** The solution is unstable. **961.** $x = 0$ is unstable, $x = t^4 + 1$ is stable. **962.** $x = t$ is stable, $x = e^t$ is unstable. **963.** $x = t$ and $x = -t$ are stable when $t < 0$ and $t > 0$ respectively. **964.** $x = 0$ is stable when $t < 0$. **965.** $x = t$ is stable, $x = e^{t^2+1}$ is unstable. **966.** The solution is unstable. **967.** The solution is unstable.

APPENDIX 1

Some formulas from differential geometry

In Cartesian coordinates (Fig. 65)

$y' = \tan\alpha.$

1. Subtangent $TP = \frac{y}{y'}$.
2. Subnormal $PN = yy'$.
3. Tangent segment length $TM = \left|\frac{y}{y'}\sqrt{1+y'^2}\right|$
4. Normal segment length $MN = \left|y\sqrt{1+x'^2}\right|$.

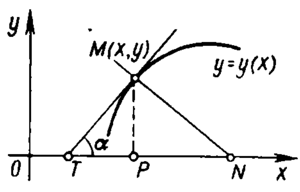

Fig. 65

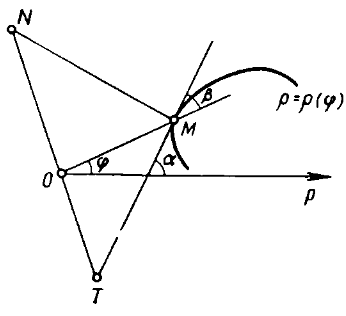

Fig. 66

5. Differential of the arc length of a curve $dS = \sqrt{dx^2 + dy^2}$. *In polar coordinates* (Fig. 66)

$\rho = \rho(\varphi), \qquad \rho' = \frac{d\rho}{d\varphi}$.

6. Angle β between a tangent and a radius vector $\tan\beta = \frac{\rho}{\rho'}$.
7. Polar subtangent $TO = \frac{\rho^2}{\rho'}$.
8. Polar subnormal $ON = \rho'$.
9. Tangent segment length $TM = \left|\frac{\rho}{\rho'} \sqrt{\rho^2 + \rho'^2}\right|$.
10. Normal segment length $MN = \sqrt{\rho^2 + \rho'^2}$.
11. Differential of the arc length of a curve $dS = \sqrt{\rho^2 + \rho'^2}\, d\varphi$.

APPENDIX 2

Basic originals and their transforms

No.	Original $f(t)$	Transform $F(p)=\int_0^\infty f(t)\,e^{-pt}dt$
1	1	$\frac{1}{p}$
2	$t^n\ (n=1,\ 2,\ \ldots)$	$\frac{n!}{p^{n+1}}$
3	$t^\alpha\ (\alpha>-1)$	$\frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$
4	$e^{\lambda t}$	$\frac{1}{p-\lambda}$
5	$\sin\omega t$	$\frac{\omega}{p^2+\omega^2}$
6	$\cos\omega t$	$\frac{p}{p^2+\omega^2}$
7	$\sinh\omega t$	$\frac{\omega}{p^2-\omega^2}$
8	$\cosh\omega t$	$\frac{p}{p^2-\omega^2}$
9	$\sin(t-\alpha)\ (\alpha>0)$	$\frac{1}{p^2+1}\,e^{-\alpha p}$
10	$\cos(t-\alpha)\ (\alpha>0)$	$\frac{p}{p^2+1}\,e^{-\alpha p}$
11	$t^n e^{\lambda t},\ n=1,\ 2,\ \ldots$	$\frac{n!}{(p-\lambda)^{n+1}}$
12	$t^\alpha e^{\lambda t}\ (\alpha>-1)$	$\frac{\Gamma(\alpha+1)}{(p-\lambda)^{\alpha+1}}$

(continued)

No.	Original $f(t)$	Transform $F(p)=\int_0^\infty f(t)\,e^{-pt}dt$
13	$e^{\lambda t}\sin\omega t$	$\frac{\omega}{(p-\lambda)^2+\omega^2}$
14	$e^{\lambda t}\cos\omega t$	$\frac{p-\lambda}{(p-\lambda)^2+\omega^2}$
15	$t\sin\omega t$	$\frac{2p\omega}{(p^2+\omega^2)^2}$
16	$t\cos\omega t$	$\frac{p^2-\omega^2}{(p^2+\omega^2)^2}$
17	$J_n(t),\ n=1,\ 2,\ \ldots$	$\frac{(\sqrt{p^2+1}-p)^n}{\sqrt{p^2+1}}$
18	$\mathrm{si}\, t$	$\frac{\operatorname{arc\,cot} p}{p}$
19	$\mathrm{Erf}\left(\frac{\alpha}{2\sqrt{t}}\right)\ (\alpha>0)$	$\frac{e^{-\alpha\sqrt{p}}}{p}$
20	$\ln t$	$\frac{1}{p}\left(\ln\frac{1}{p}-c\right)$, $c=0.57722$ being the Euler constant

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